

# NELINEARNE OSCILACIJE

Livija CVETIĆANIN

# Sadržaj

- 1. Uvod
- 2. Strogo nelinearne oscilacije – analitičko rešenje
- 3. Oscilator sa nelinearnim prigušenjem
- 4. Analitičke metode rešavanja oscilovanja rotora
- 5. Elastični metamaterijali
- 6. Deterministički kaos



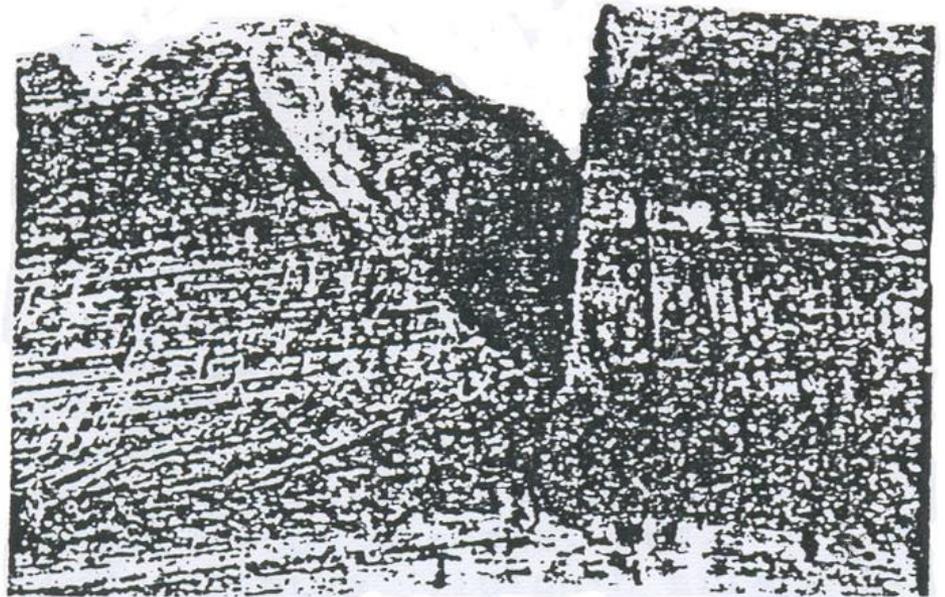
TACOMA.AVI

# Zahtevi savremene masinogradnje:

- Velika produktivnost (povećana radna brzina)
- Ekonomičnost (manji utrošak energije i materijala)
- Visok kvalitet proizvoda
- Ispunjenje kriterijuma zaštite radne i životne sredine

# Posledice vibracija

- Smanjenje radnog veka mašine
- Smanjena tačnost
- Pojava havarija i lomova
- Pojava buke



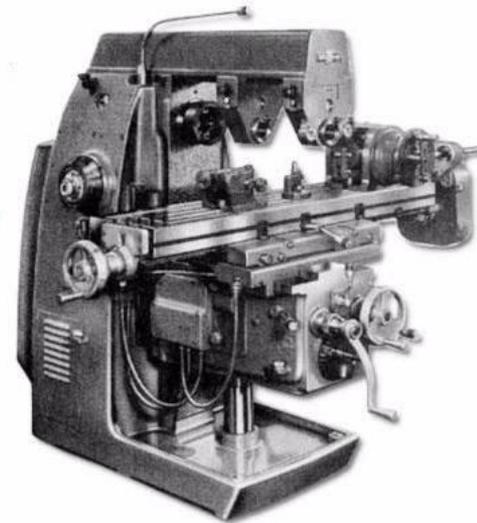
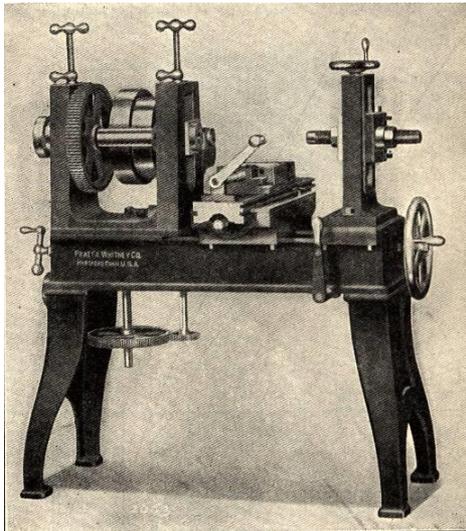
- -Vibracije kao mera radne sposobnosti
- -Vibrodijagnostika stanja

# Otklanjanje vibracija

- Projektovanjem
- Konstruisanjem
- Pravilnom izradom dela
- Pravilnom montažom
- Pravilnim rukovanjem mašinom

# Projektovanje i proračun

- Statički proračun
- Predviđanje korišćenjem linearnog modela

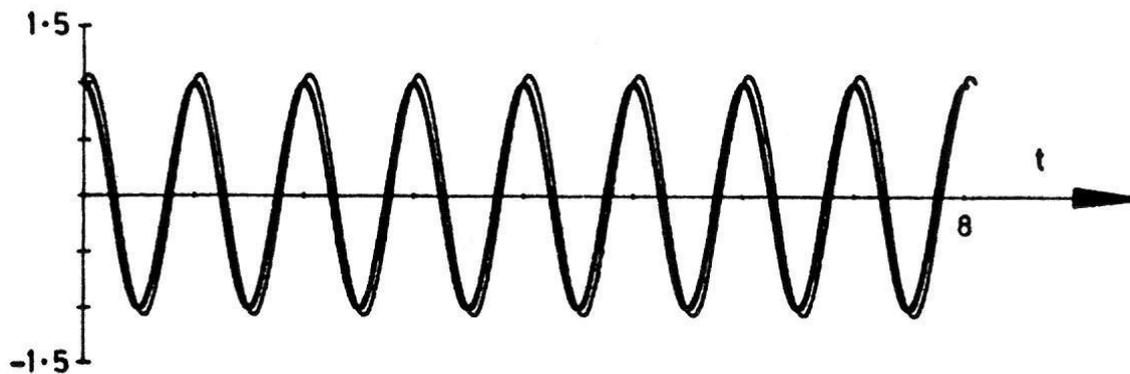


$$\ddot{u} + \omega_0^2 u = 0$$

# Periodično kretanje

## Harmonijske oscilacije

$$\ddot{u} + \omega_0^2 u = 0$$

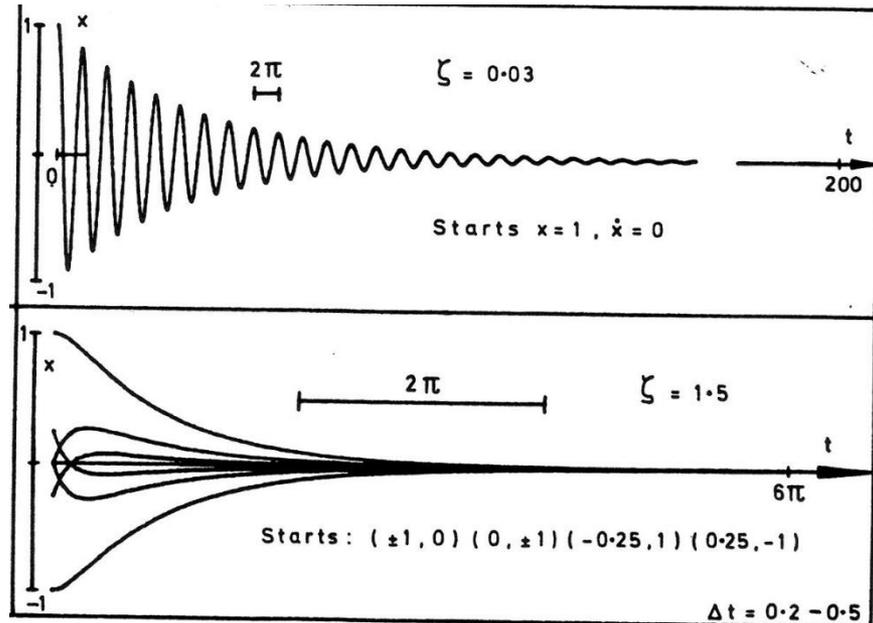


# Prigušene oscilacije

$$u + \omega_0 u + 2\mu \dot{u} = 0$$

$$\ddot{u} + \omega_0^2 u + 2\mu \dot{u} = 0$$

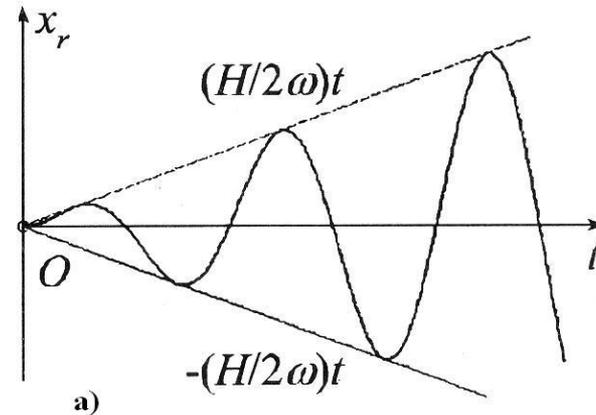
- Prigušenje je malo
- Prigušenje je veliko



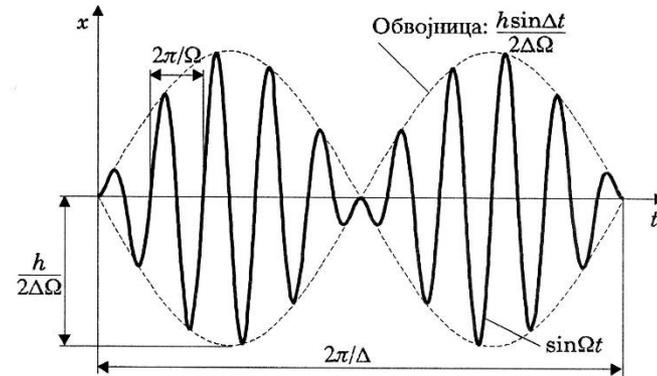
# Prinudni linearni oscilator

$$\ddot{u} + \omega_0^2 u = K \cos(\Omega t)$$

- Rezonanca



- Podrhtavanje

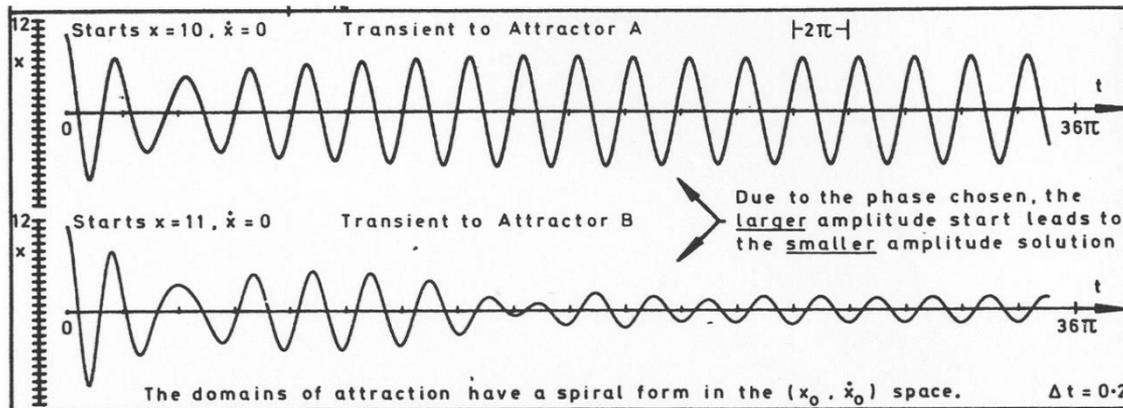
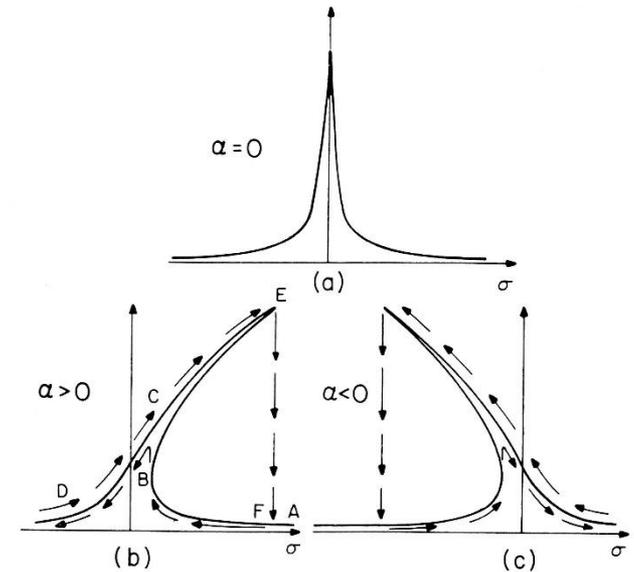


# Nelinearnosti

- Geometrijska nelinearnost
- Fizička nelinearnost
  - Vrsta materijala
  - Prigušenje: suvo trenje, viskozno trenje

# Slaba nelinearnost funkcija pomerenja

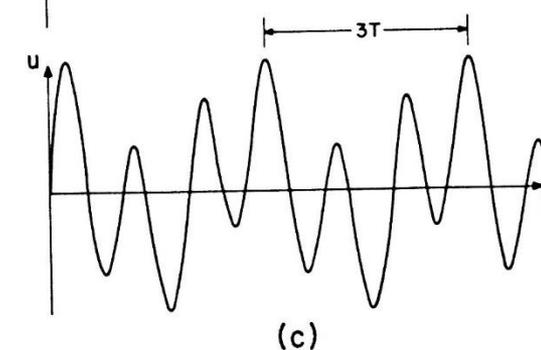
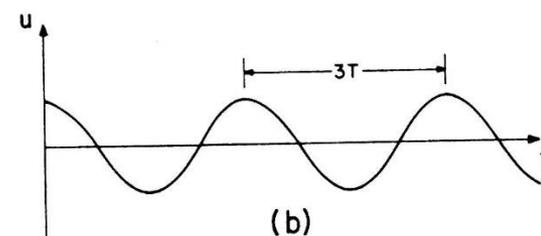
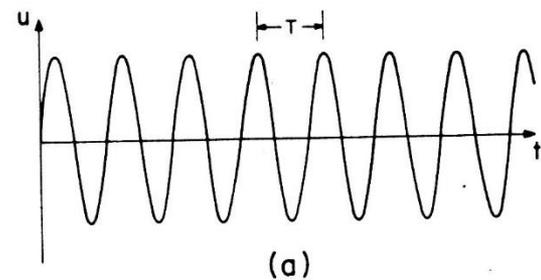
- Primarna rezonanca



$$\Omega = 3\omega_0$$

# Sekundarna rezonanca

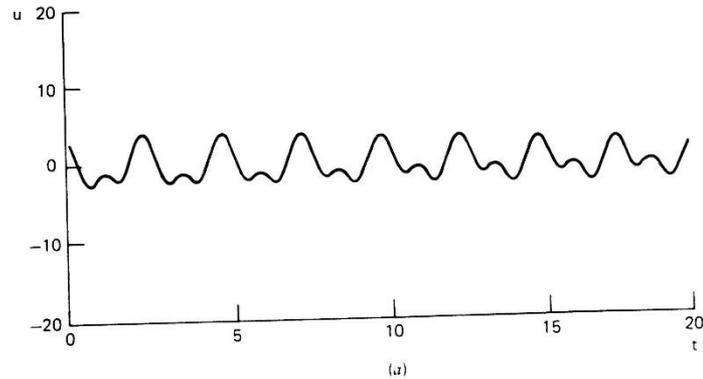
- Subharmonijska rezonanca
- Superharmonijska rezonanca



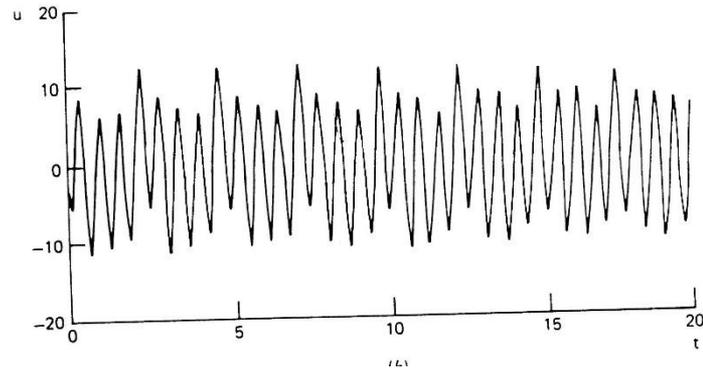
# Višefrekventna pobuda

$$\ddot{u} + \omega_0^2 u = -2\varepsilon\mu\dot{u} - \varepsilon\alpha u^3 + K_1 \cos(\Omega_1 t + \theta_1) + K_2 \cos(\Omega_2 t + \theta_2)$$

- Linearan model

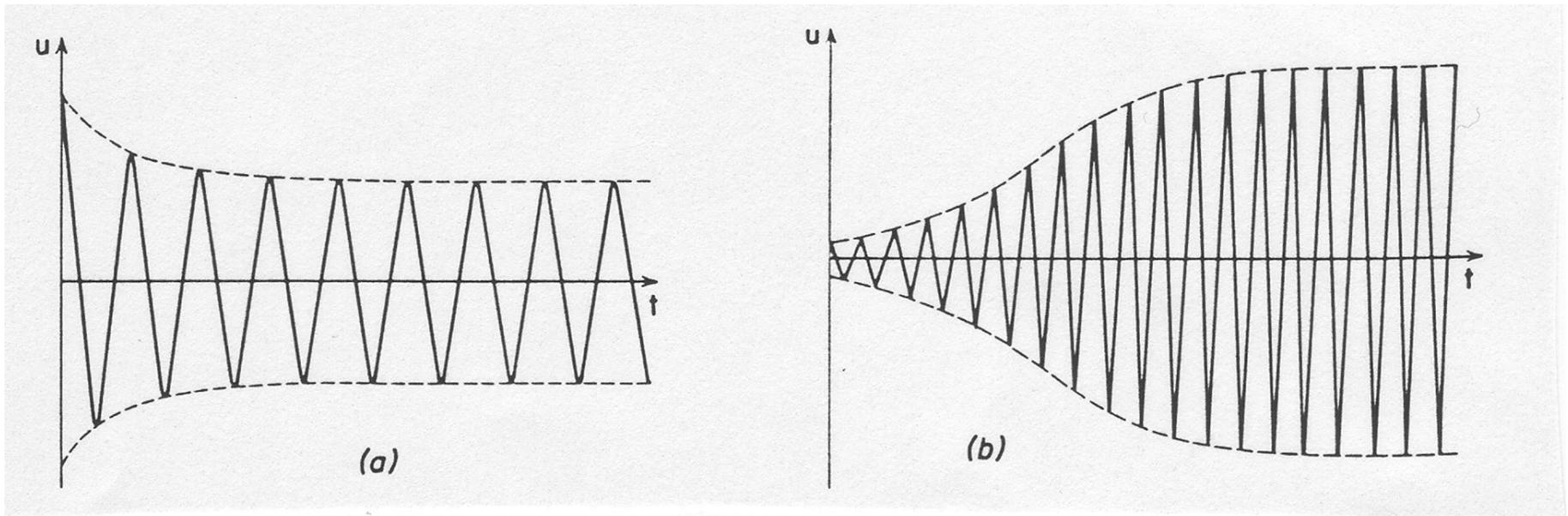


- Nelinearan model



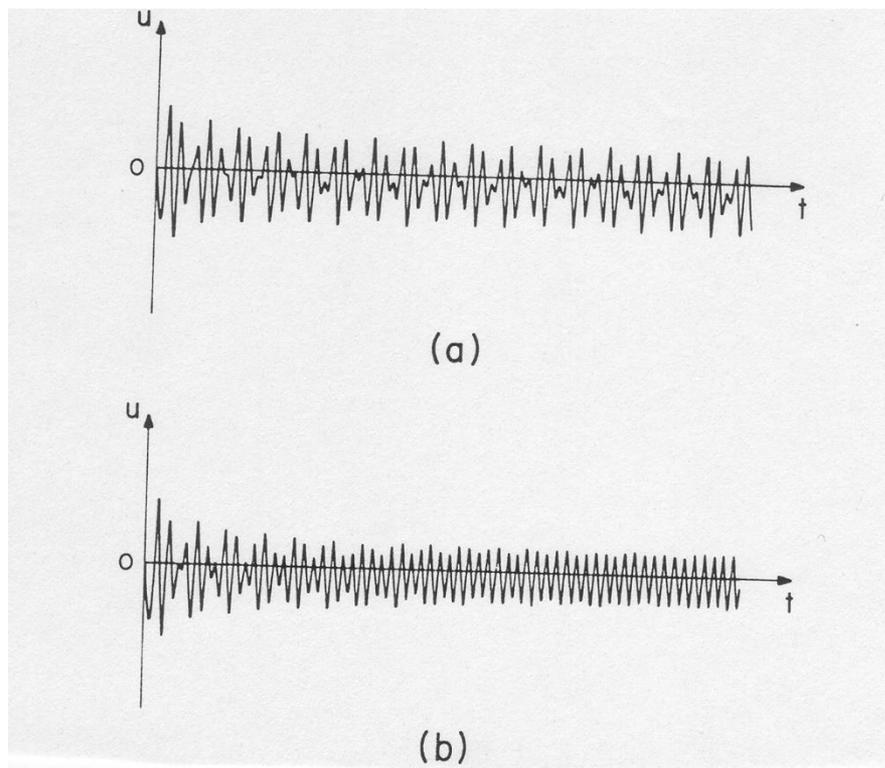
# Slaba nelinearnost funkcija brzine Samopobudne oscilacije

Van der Polov oscilator



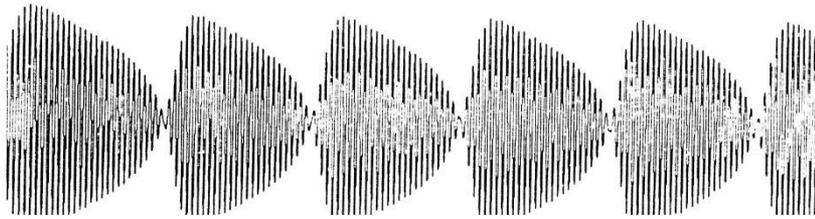
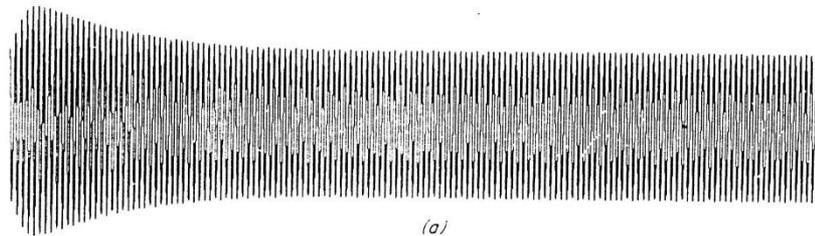
# Prinudni nelinearni oscilator

Uticaj amplitude  
prinudne sile  
“suzbijanje”



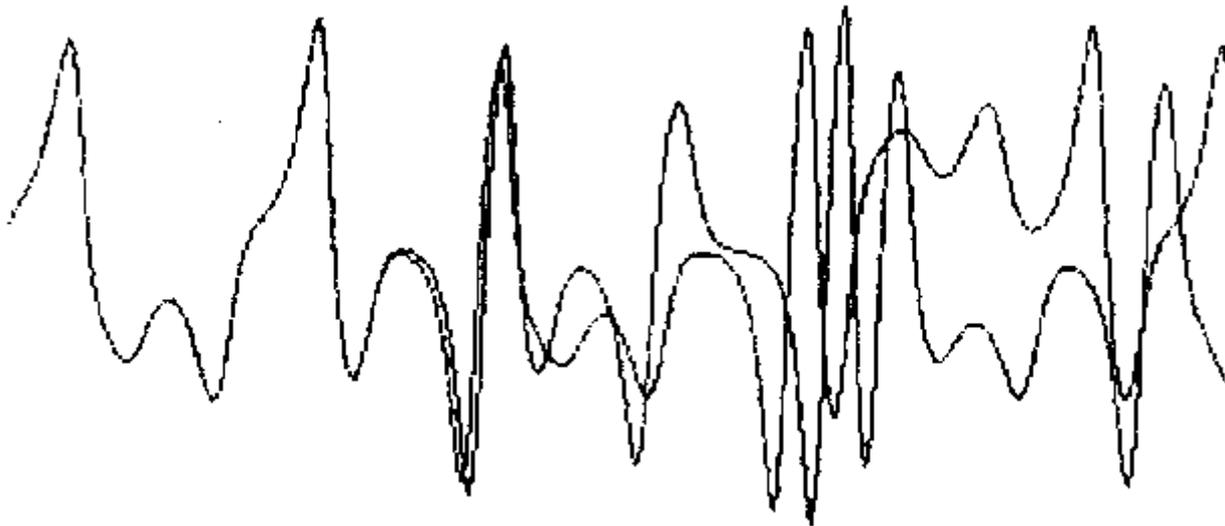
# Prinudni nelinearni oscilator

- Sinhronizacija
- Bijenje



# Strogo nelinearni oscilator

## “Buterfly Effect”



*Figure 1: Lorenz's experiment: the difference between the start of these curves is only .000127. (Ian Stewart, Does God Play Dice? The Mathematics of Chaos, pg. 141)*

# Haotično kretanje

- Edward Lorenz (1961)
  - butterfly effect
- Yoshi Ueda (1961)-(1970)
- T.Y. Li James, A. Zorke, Period three implies chaos, Amer. Math. Monthly 82, 985, 1975.

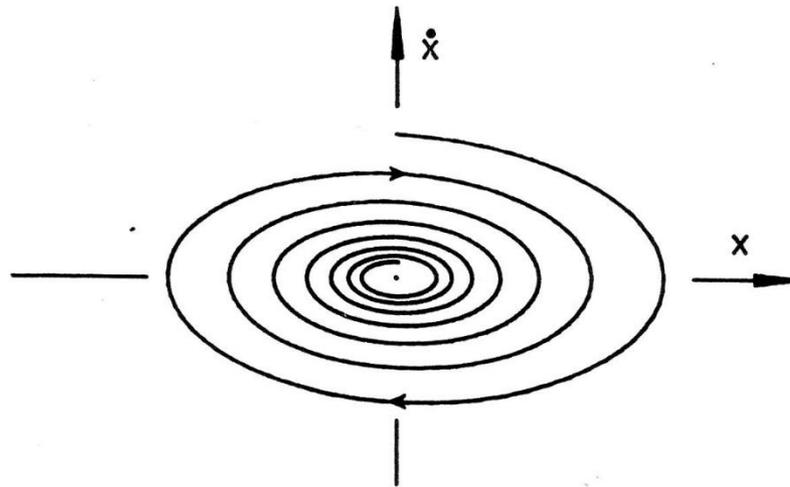
# Haos

- Χαος (grčki) – praznina, prazan prostor
- Grčka mitologija

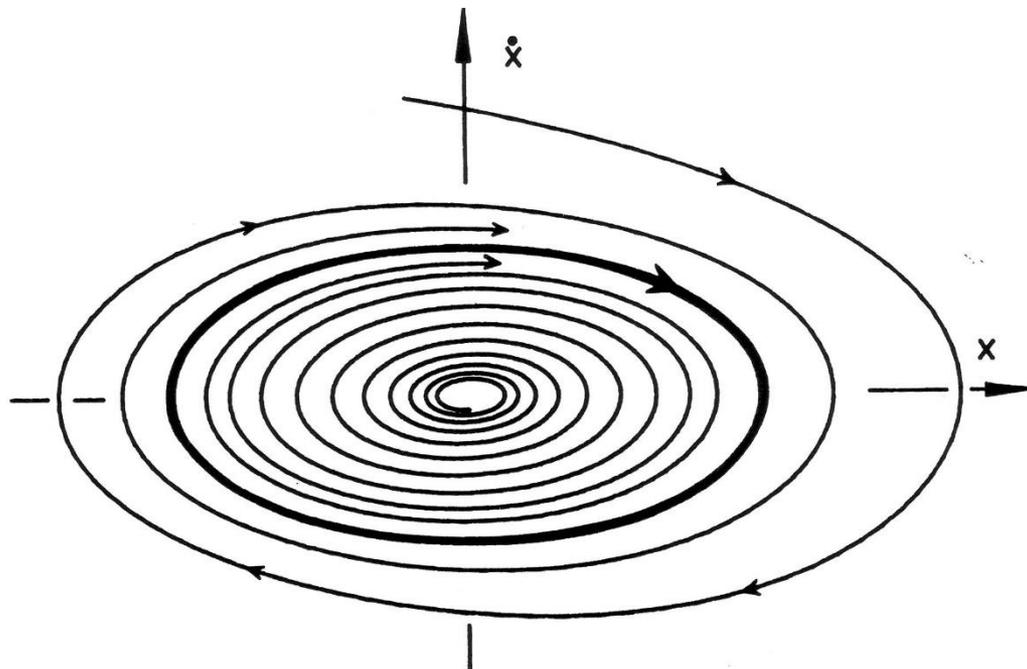


# Atraktori

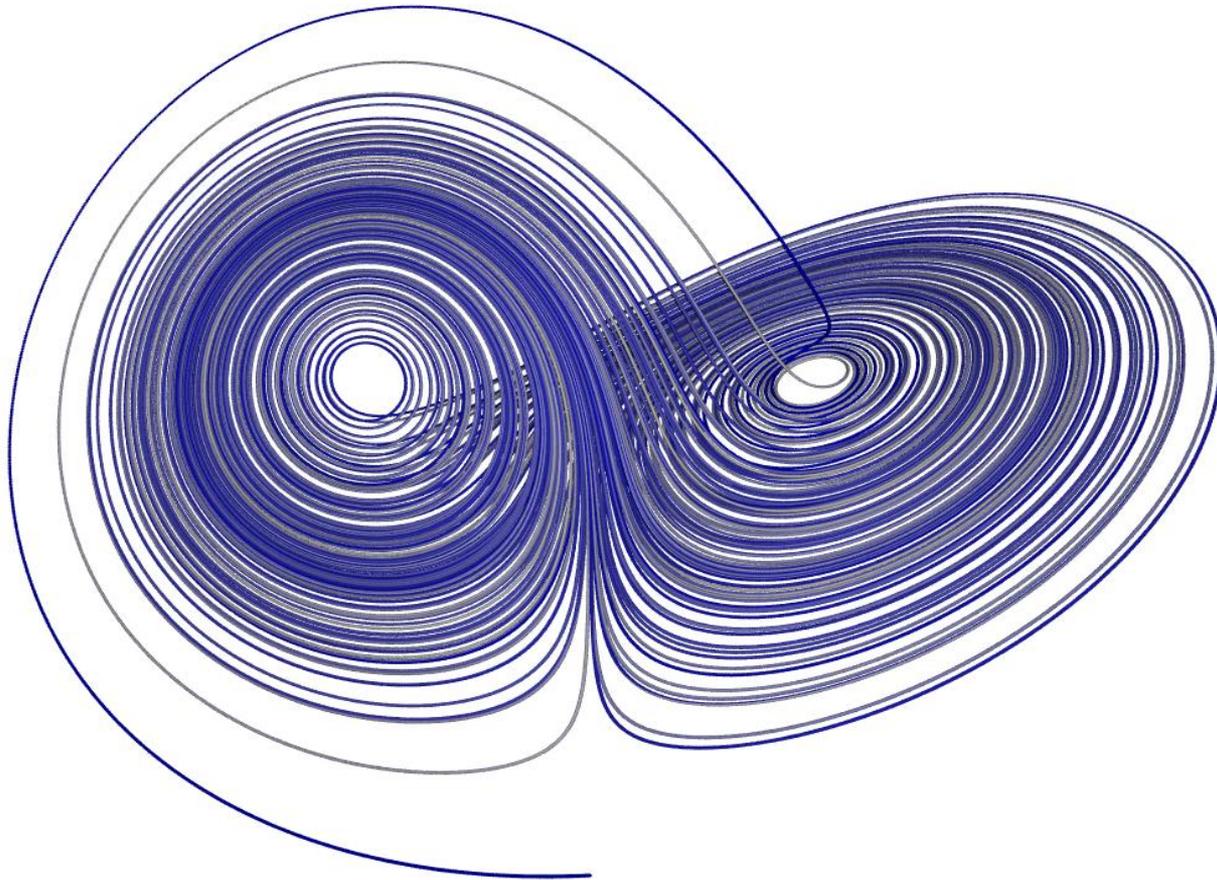
- Tačkasti atraktor

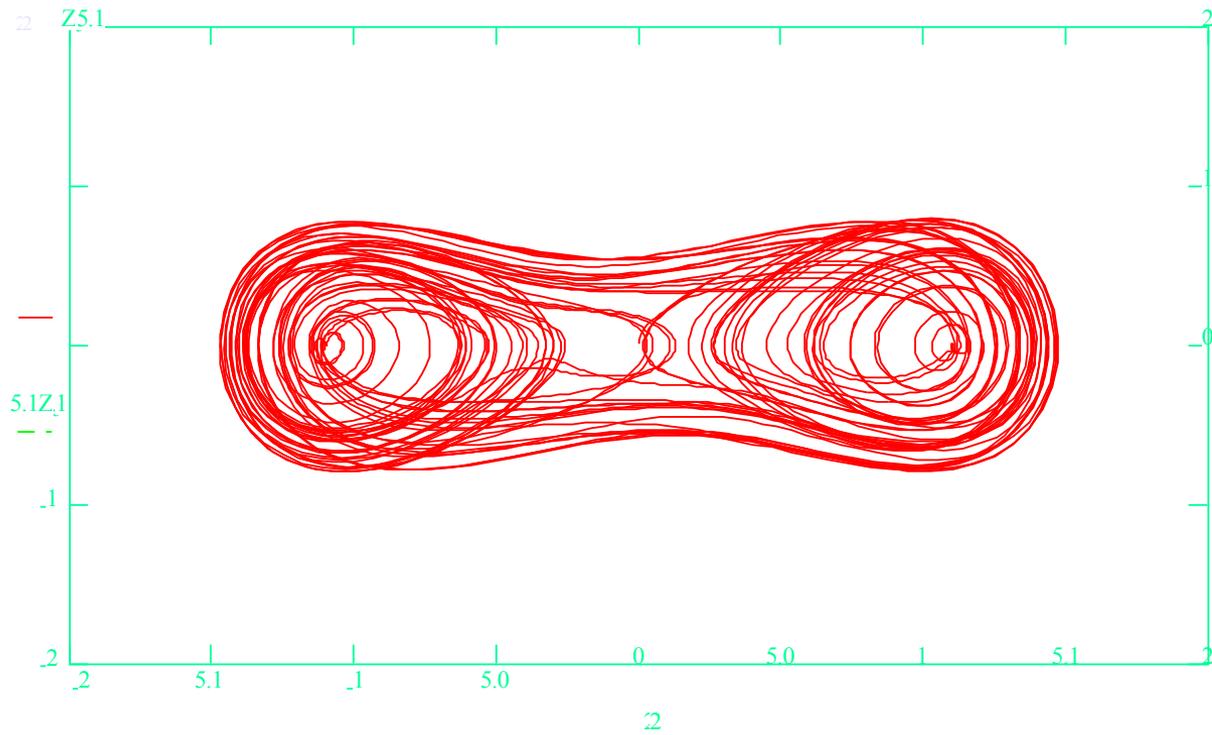
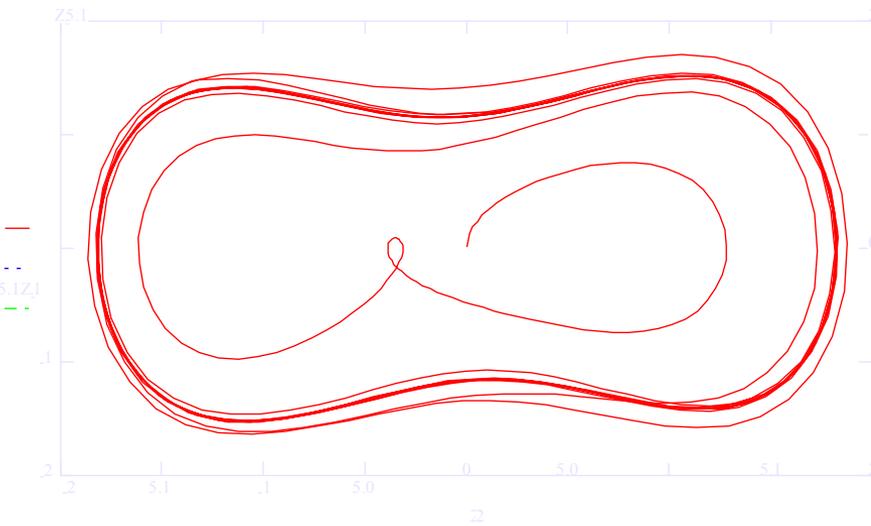
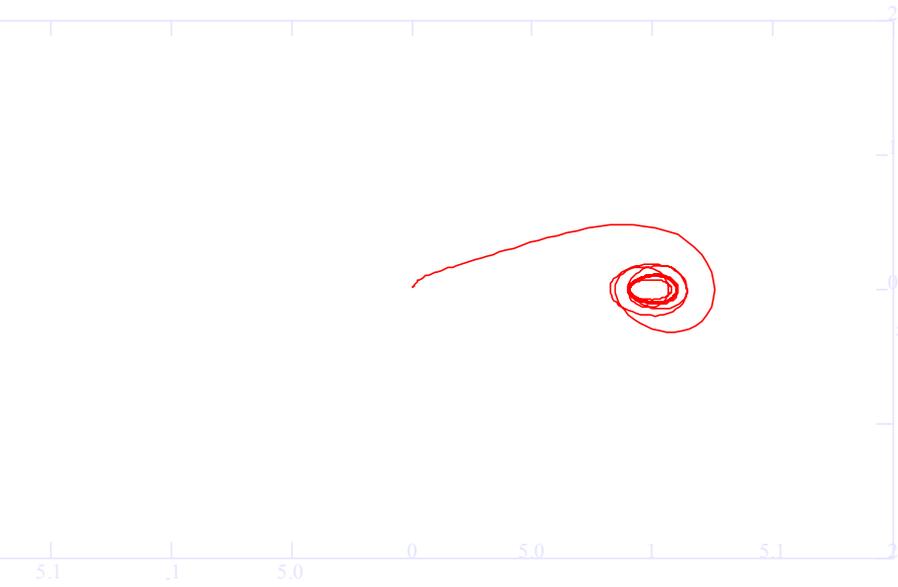


# Ciklični atraktor



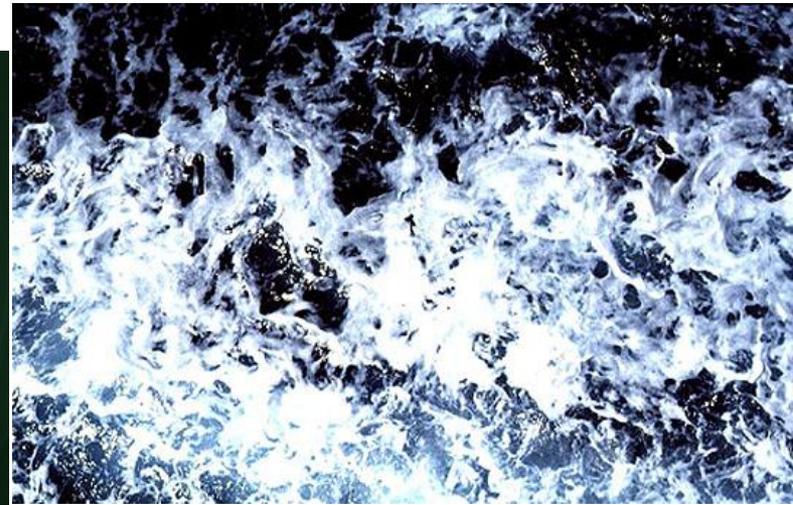
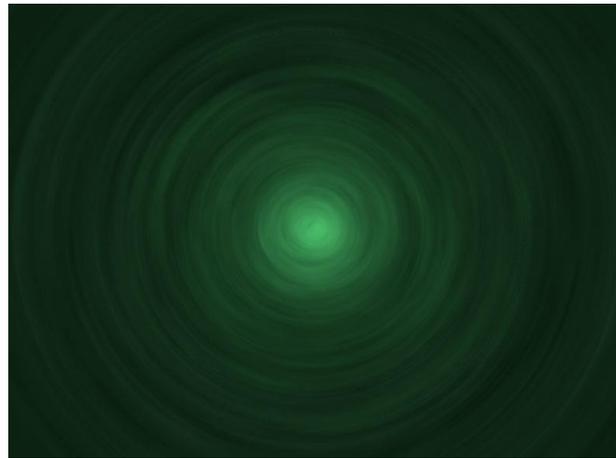
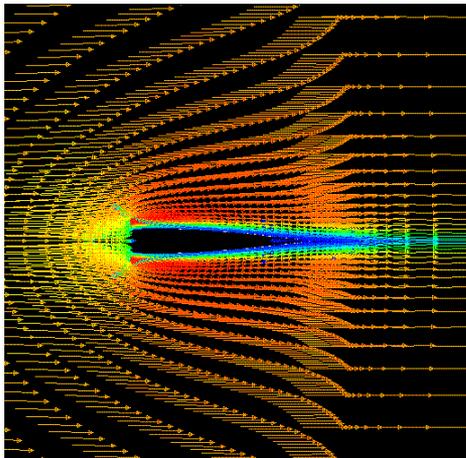
# Haotični atraktor





# Bifurkacije

- Postepene
- Skokovite - katastrofalne



# Zaključak

- Kompleksno dinamičko modeliranje mašinskog dela
- Pravilna simulacija
- Mogućnost predviđanja kretanja

# Primenljivost teorije

- Analiza kretanja populacije
- Istraživanja u oblasti neurologije i reakcija mozga
- Dijagnostika rada srca
- Predviđanje oko promene krvnog pritiska
- Primena u hemiji
- Nebeska mehanika
- Akceleratori i proizvodnja nano čestica

**2. STROGO NELINEARNE  
OSCILACIJE –  
Analitička rešenja**

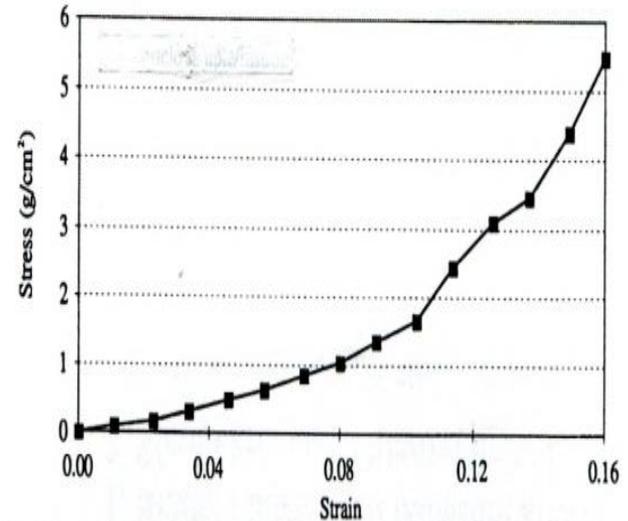
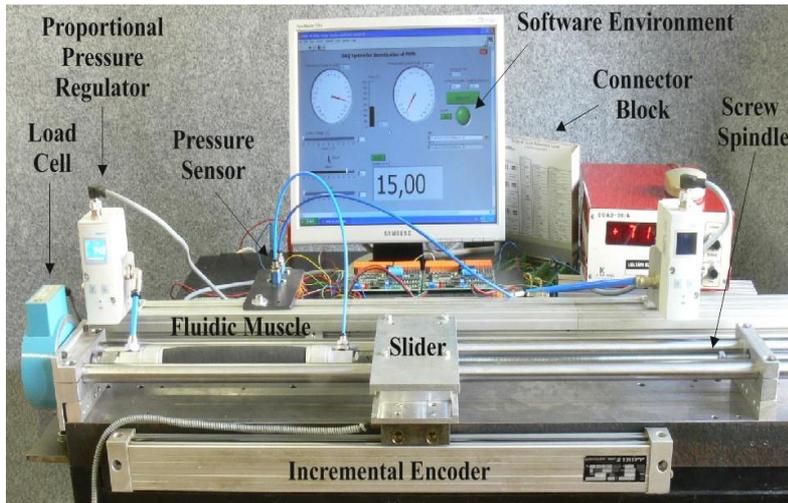
# Sadržaj

- Pojam strogo nelinearnog oscilatora
- Generalizacija metode Krilov-Bogoljubov
- Primeri primene na sisteme sa jednim, dva i beskonačno mnogo stepeni slobode
- Zaključak

# Uvod

- Oscilator
- Modeliranje
- Linearni oscilator (period, frekvencija i amplituda)

# Nelinearni reološki model



$$\sigma = E \varepsilon^{\alpha}$$

# Začetnici teorije nelinearnih oscilacija

**Nikolaj Mitrofanovič Krilov**



29.11.1879. Petersburg, Rusija  
- 11.5.1955. Moskva, SSSR

**Nikolaj Nikolajevič Bogoljubov**



21.8.1909. Nižnji Novgorod  
(Gorki), Rusija  
13.2.1992. Moskva, Rusija

# Polje zajedničkog rada

- Od 1930. godine rade zajedno na problemima Nelinearna mehanika i matematička fizike,
- - metode asimptotske integracije nelinearnih jednačina koje modeliraju oscilatorne sisteme.

# Šta je bilo novo?

- Umesto da samo dokazuju egzistenciju rešenja, oni su **sračunavali rešenja, odn., određivali:**
- Približna periodična rešenja.

# Procedure sa "malim" parametrima

- Metode za asimptotsko rešavanje nelinearnih jednačina sa " malim " parametrom koji opisuju oscilatorne procese

# Princip naučnog rada 'Kievške škole'

- Globalno oceniti karakter fizičkog problema
- Dokazati da je problem rešiv, i tada, bez obzira na poteškoće
- Razraditi adekvatan matematički aparat za rešavanje problema (tako stižemo do Davida Hilberta "**Wir müssen wissen, wir werden wissen**").

# Metod Krilov – Bogoljubov za strogo nelinearne oscilatore

- 

$$\ddot{x} + c_{\alpha}^2 x |x|^{\alpha-1} = \varepsilon f(x, \dot{x})$$

- Generating equation

- $$\ddot{x} + c_{\alpha}^2 x |x|^{\alpha-1} = 0.$$

# Opšte rešenje

- $x = aca(\alpha, 1, \psi),$
- Ateb function
- 
- $\psi = \omega t + \theta,$

- $\frac{d}{d\psi} ca(\alpha, 1, \psi) = -\frac{2}{1+\alpha} sa(1, \alpha, \psi),$   
 $\frac{d}{d\psi} sa(1, \alpha, \psi) = ca^\alpha(\alpha, 1, \psi),$

- $\dot{x} = -\frac{2a\omega}{1+\alpha} sa(1, \alpha, \psi),$   
 $\ddot{x} = -\frac{2a\omega^2}{1+\alpha} ca^\alpha(1, \alpha, \psi).$

$$\omega = \sqrt{c_\alpha^2 \frac{1+\alpha}{2} a^{(\alpha-1)/2}}.$$

# Približno rešenje za nelinearan oscilator

- $x = a(t)ca(\alpha, 1, \psi),$
- $\dot{x} = -\frac{2a\omega}{1+\alpha}sa(1, \alpha, \psi),$

With

- $\dot{\psi} = \omega + \dot{\theta}(t),$
- $\omega = \sqrt{c_\alpha^2 \frac{1+\alpha}{2} a^{(\alpha-1)/2}}.$

# Ograničenje za prvi izvod

- $\dot{x} = \dot{a}ca(\alpha, 1, \psi) - \frac{2a\omega}{1+\alpha}sa(1, \alpha, \psi) - \frac{2a\dot{\theta}}{1+\alpha}sa(1, \alpha, \psi)$
- Constraint
- $\dot{a}ca(\alpha, 1, \psi) - \frac{2a\dot{\theta}}{1+\alpha}sa(1, \alpha, \psi) = 0. \quad (1)$

- $$\begin{aligned}
 & -\dot{a}\omega sa(1, \alpha, \psi) - \frac{2a\omega\dot{\theta}}{1+\alpha} ca^\alpha(1, \alpha, \psi) = \\
 & \varepsilon f \left( aca(1, \alpha, \psi), -\frac{2a\omega}{1+\alpha} sa(1, \alpha, \psi) \right)
 \end{aligned}$$

(2)

# Jednačina oscilovanja u novim promenljivama

$$\dot{a} = \frac{1}{\omega} \varepsilon f \left( \begin{array}{c} aca(1, \alpha, \psi), -\frac{2a\omega}{1+\alpha} sa(1, \alpha, \psi) \\ sa(1, \alpha, \psi) \end{array} \right)$$

$$a\dot{\theta} = -\frac{\alpha+1}{2\omega} \varepsilon f \left( \begin{array}{c} aca(1, \alpha, \psi), -\frac{2a\omega}{1+\alpha} sa(1, \alpha, \psi) \\ ca(1, \alpha, \psi) \end{array} \right)$$

# Osrednjavanje po periodu

- $T_\alpha = 2\Pi_\alpha = 2B \left( \frac{1}{\alpha+1}, \frac{1}{2} \right),$
- $a\dot{\theta} = \frac{-(\alpha+1)}{2\omega} \frac{1}{2\Pi_\alpha} \int_0^{2\Pi_\alpha} \varepsilon f \left( aca, -\frac{2a\omega}{1+\alpha} sa \right) cad\psi$
- $\dot{a} = -\frac{1}{\omega} \frac{1}{2\Pi_\alpha} \int_0^{2\Pi_\alpha} \varepsilon f \left( aca, -\frac{2a\omega}{1+\alpha} sa \right) sad\psi$

Strogo nelinearan oscilator  
poremećen malom linearnom  
silom koja zavisi od položaja  $x$

- $\ddot{x} + c_\alpha^2 x |x|^{\alpha-1} = \varepsilon c_1 x,$
- Averaged equations
- $\dot{a} = 0, \quad a^{(\alpha-1)/2} \dot{\theta} = -\varepsilon c_1 K \sqrt{\frac{1+\alpha}{2c_\alpha^2}},$

# Amplituda i faza oscilovanja

$$a = a_0, \quad \theta = \theta_0 - \frac{\varepsilon c_1 K t}{a_0^{(\alpha-1)/2}} \sqrt{\frac{1+\alpha}{2c_\alpha^2}},$$

- $K = \frac{1}{2\Pi_\alpha} \int_0^{2\Pi_\alpha} c a^2(1, \alpha, \psi) d\psi$

- $\omega = \sqrt{c_\alpha^2 \frac{1+\alpha}{2}} a_0^{(\alpha-1)/2}$

# Približno rešenje

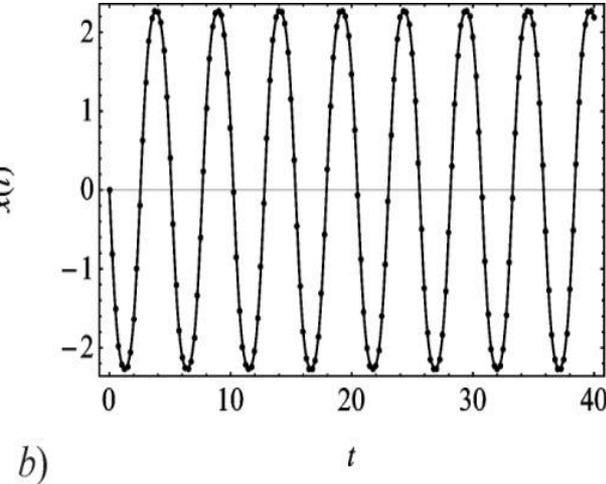
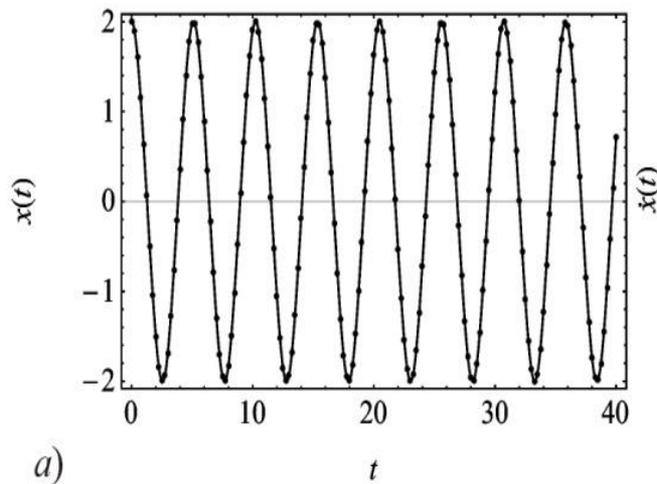
$$x = a_0 c a(\alpha, 1, \psi), \quad \dot{x} = -a_0 \omega s a(1, \alpha, \psi),$$

- where

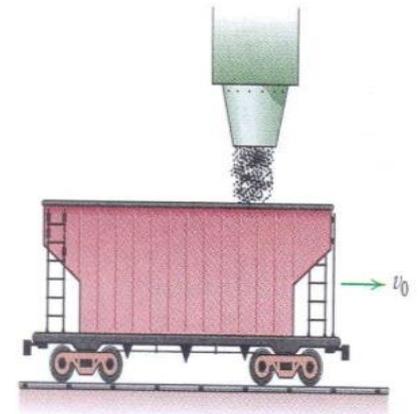
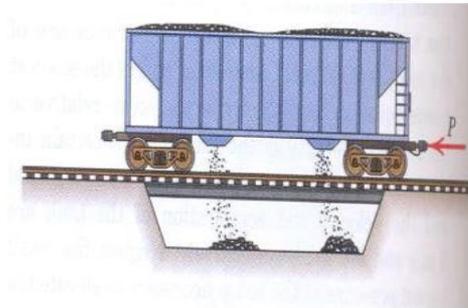
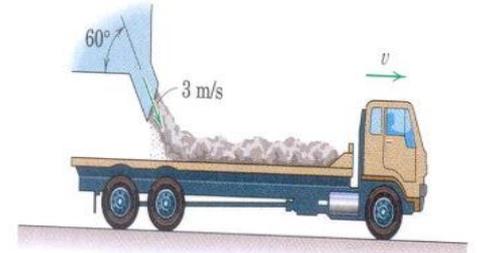
- $\psi = \theta_0 + t \left( \omega - \frac{\varepsilon c_1 K(1+\alpha)}{2\omega} \right)$

- $P = \frac{2\pi}{\omega - \frac{\varepsilon c_1 K(1+\alpha)}{2\omega}}$

# Poređenje analitičkog i numeričkog rešenja



# Promenljiva masa

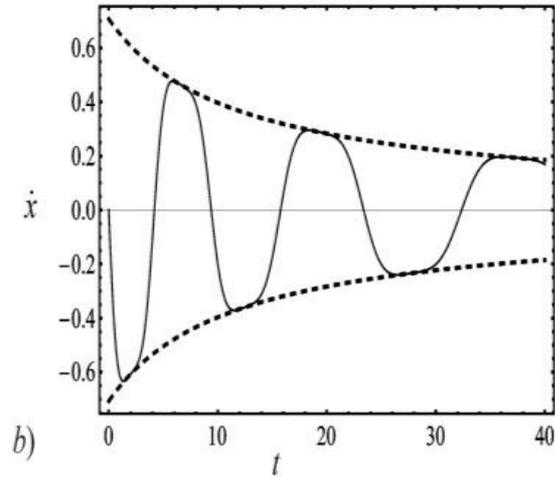
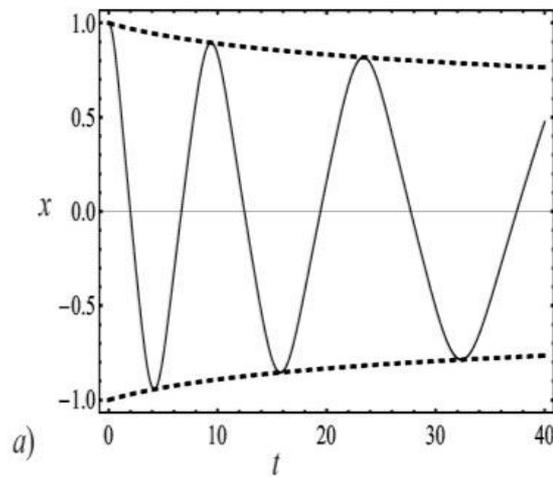


# Promenljiva masa

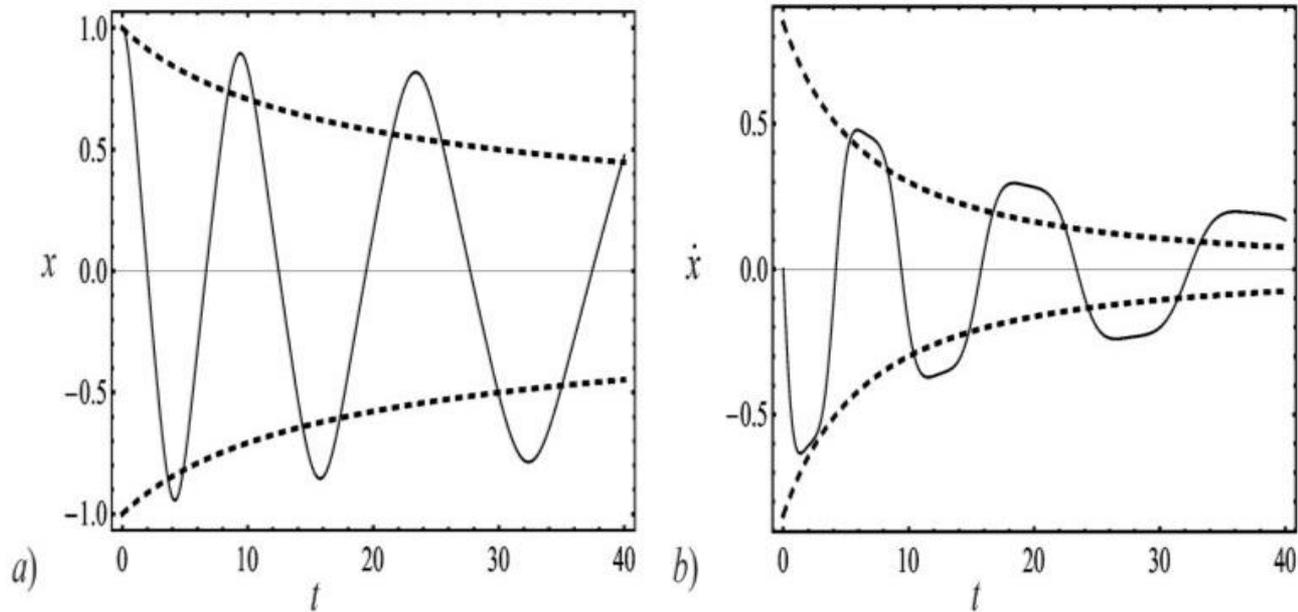
$$\frac{d}{dt}(m\dot{x}) + x^3 = 0,$$

$$m = 1 + 0.1t, \quad x(0) = x_0, \quad \dot{x}(0) = 0.$$

$$A = (1 + 0.1t)^{-0.16667}, \quad \dot{x}_{\max} A = \frac{\sqrt{2}}{2} (1 + 0.1t)^{-0.83333}.$$



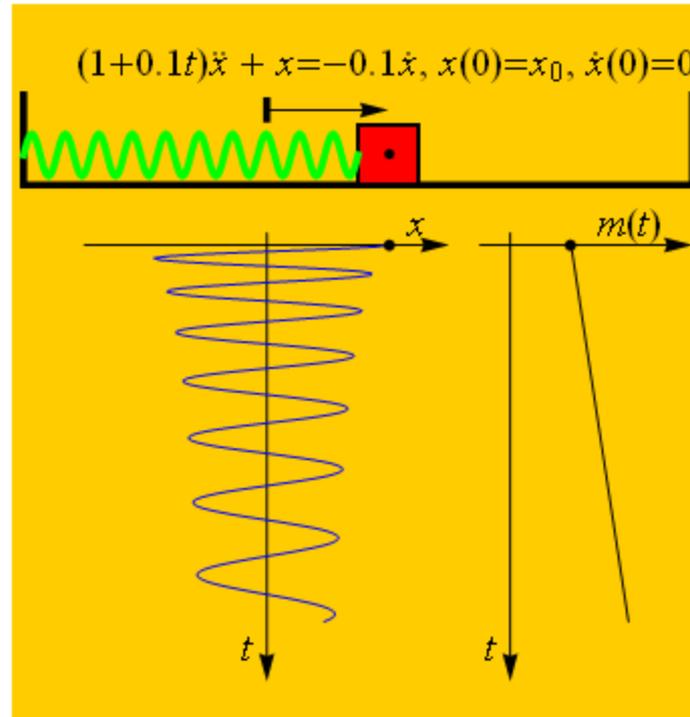
$$A = (1 + 0.1t)^{-0.5}, \quad \dot{x}_{\max A} = 0.84721(1 + 0.1t)^{-3/2}.$$



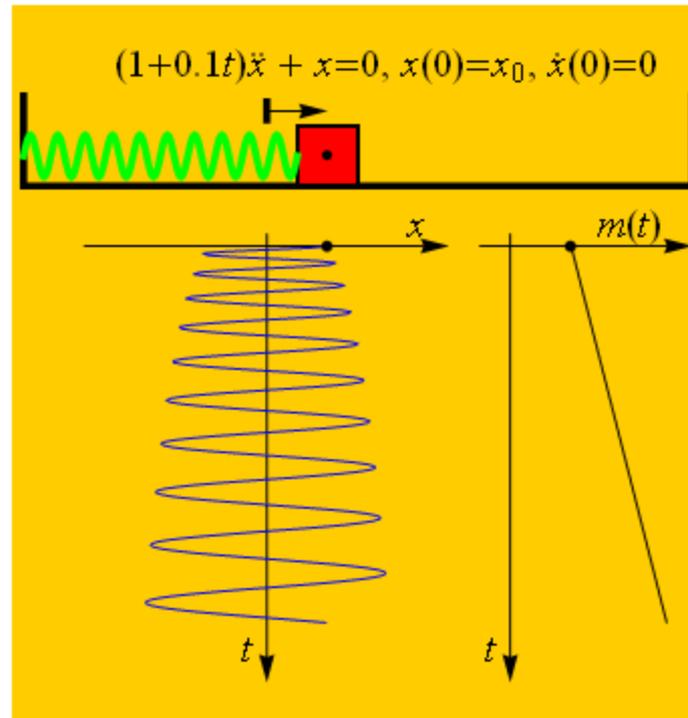
$$P\left(\left|\frac{x_m - A_m}{x_m}\right|\right) \approx 19.829\%,$$

$$P\left(\left|\frac{\dot{x}_m - \dot{x}_A}{\dot{x}_m}\right|\right) \approx 27.959\%.$$

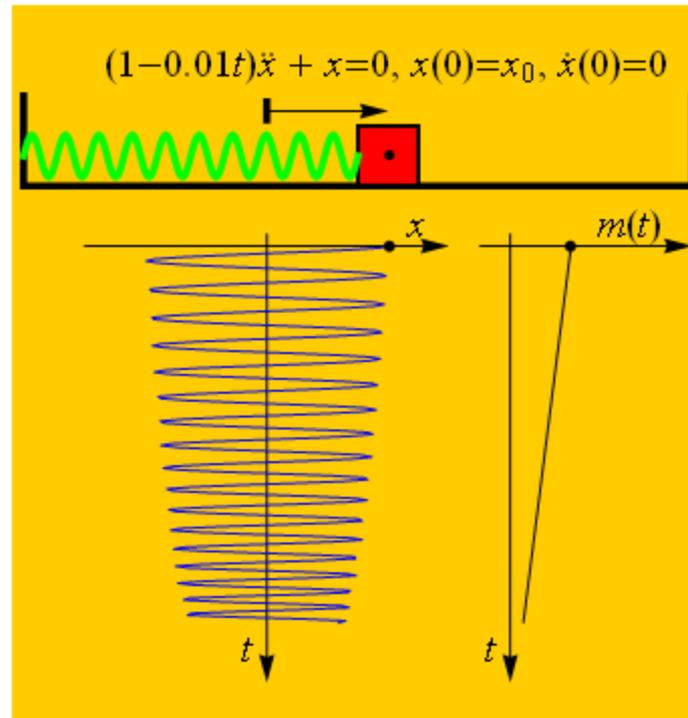
$$u=0$$



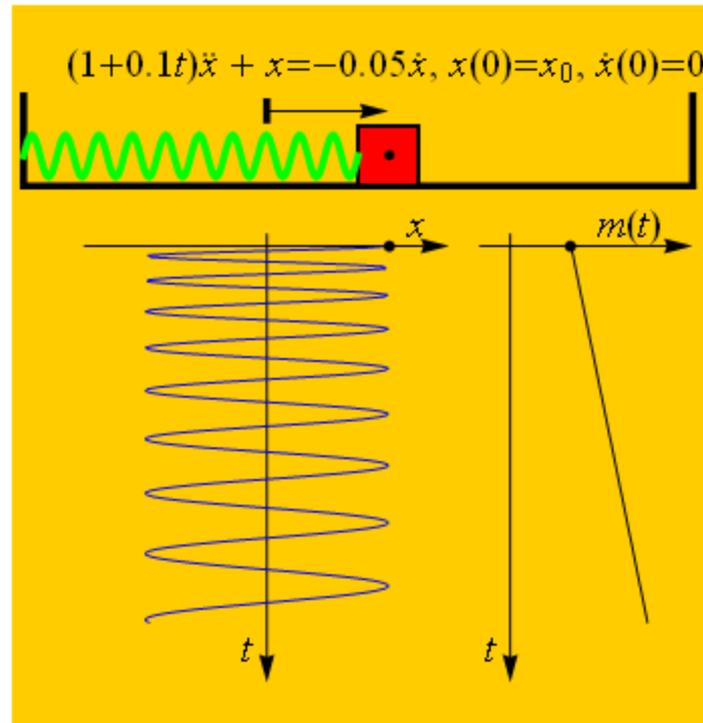
# Masa raste



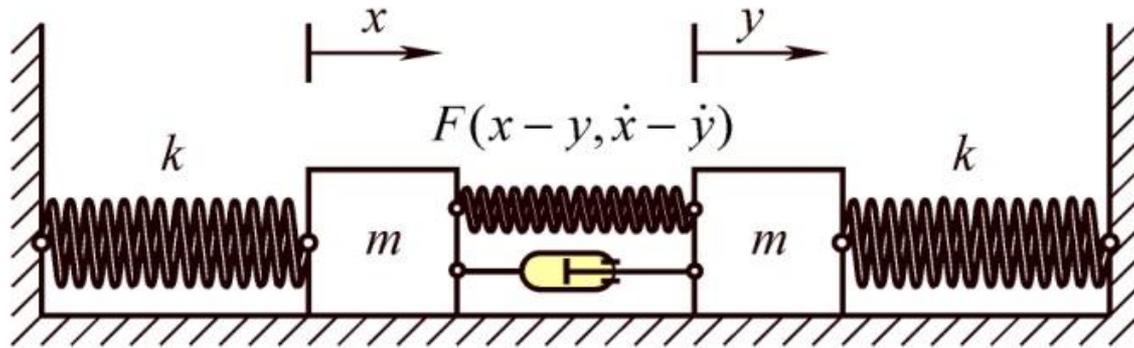
# Masa se smanjuje



$$u = 0.5\dot{x}$$



# Oscilator sa dva stepena slobode kretanja

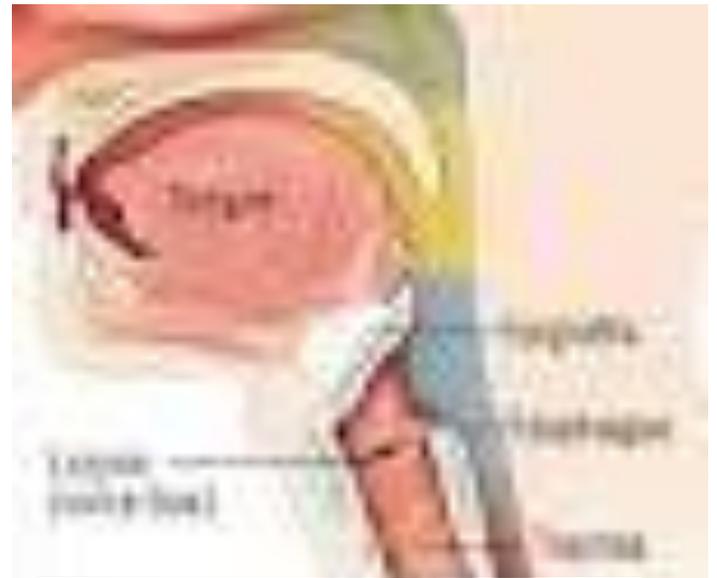
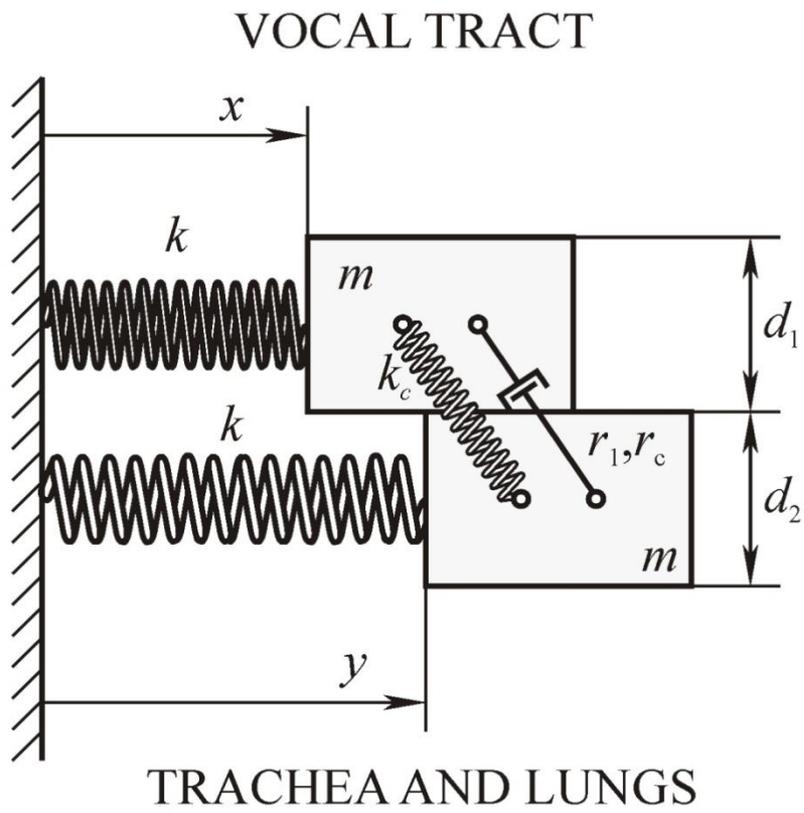


$$m\ddot{X} + kX + F(X - y, \dot{X} - \dot{y}) = 0,$$
$$m\ddot{y} + ky - F(X - y, \dot{X} - \dot{y}) = 0,$$

Početne vrednosti

- $X(0) = X_0, \quad y(0) = y_0,$
- $\dot{X}(0) = \dot{X}_0, \quad \dot{y}(0) = \dot{y}_0.$

# Glasna žica



# Nove promenljive

$$x = X - y, \quad u = X + y.$$

- $\ddot{u} + \omega_0^2 u = 0, \quad \ddot{x} + F(x, \dot{x}) = 0.$

- Početne vrednosti

$$x(0) = X_0 - y_0 = x_0, \quad u(0) = X_0 + y_0 \\ = u_0,$$

$$\dot{x}(0) = \dot{X}_0 - \dot{y}_0 = \dot{x}_0, \quad \dot{u}(0) = \dot{X}_0 + \dot{y}_0 \\ = \dot{u}_0$$

# Nelinearna sila u sloju u lamina propria

$$F = c_{\alpha}^2 (X - y)^{\alpha} + r_1 (\dot{X} - \dot{y}) - r_{\beta} (\dot{X} - \dot{y}) |\dot{X} - \dot{y}|^{\beta-1}$$

vagyis

$$F = c_{\alpha}^2 x^{\alpha} + r_1 \dot{x} - r_{\beta} \dot{x} |\dot{x}|^{\beta-1}$$

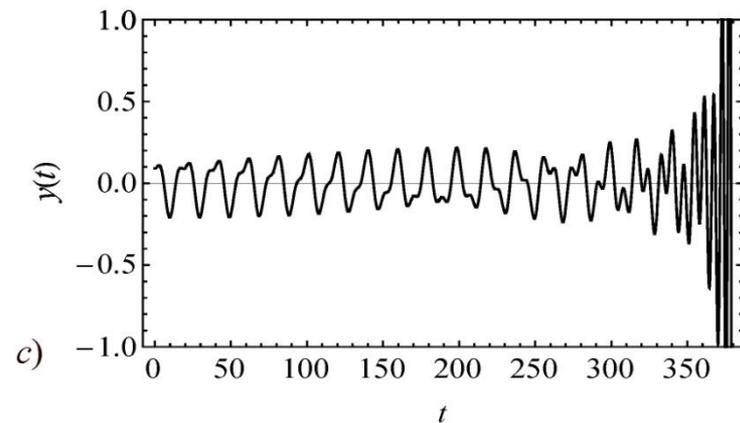
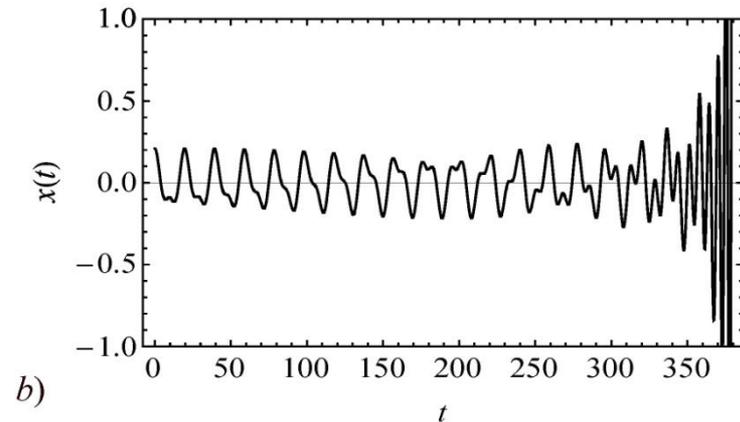
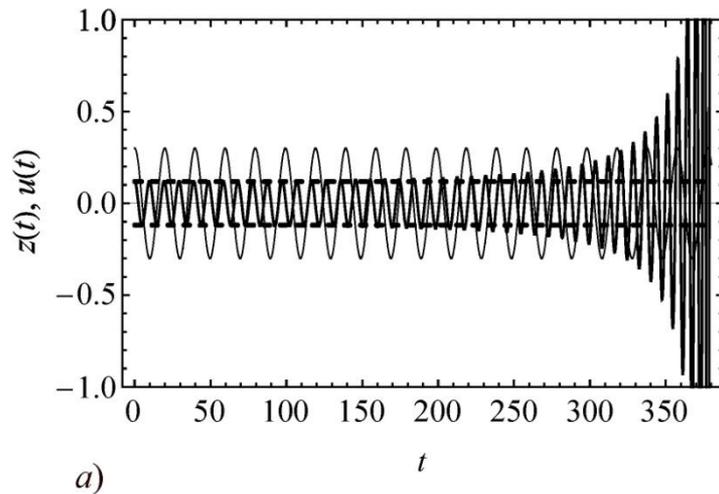
- Cornjaeger, R. Model of the sound generation in a human larynx, PhD thesis, Braunschweig, 1978.

# Razdvojena nelinearna jednačina oscilovanja

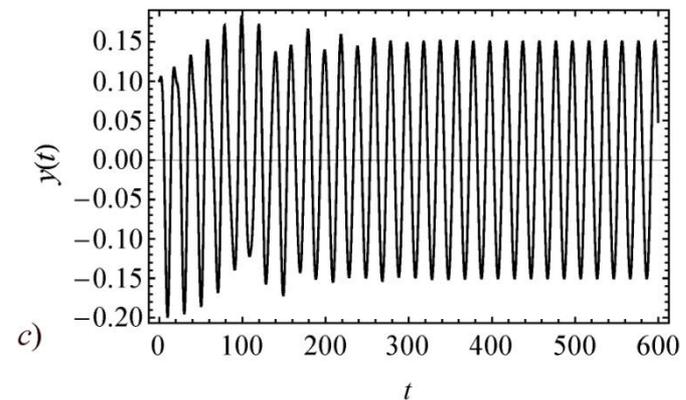
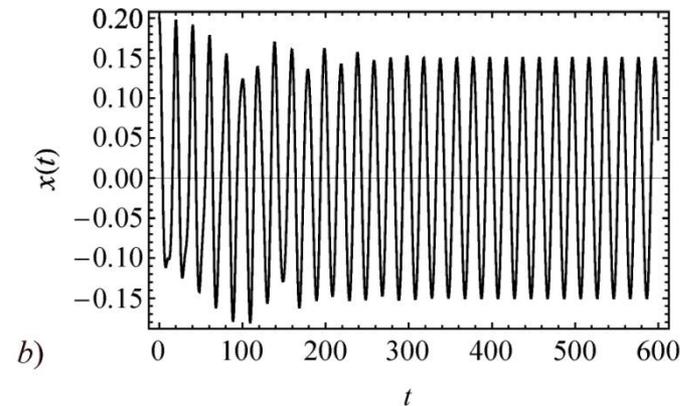
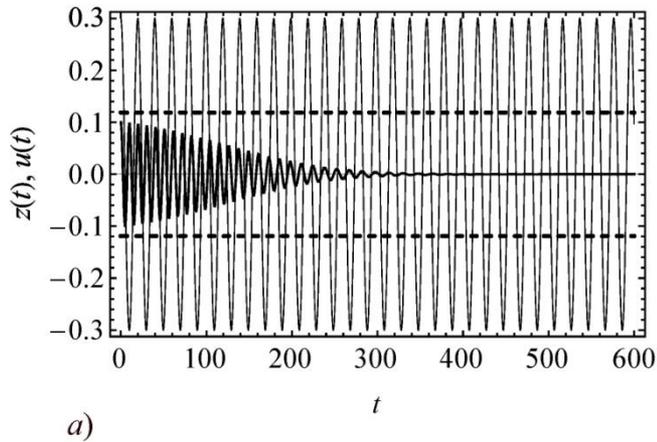
$$\ddot{x} + c_{\alpha}^2 x |x|^{\alpha-1} = \varepsilon f(x, \dot{x})$$

- Rešenje sistema jednačina
- $X = \frac{u_0}{2} \cos(\omega_0 t) + \frac{a_0}{2} ca(\alpha, 1, \psi)$
- $y = \frac{u_0}{2} \cos(\omega_0 t) - \frac{a_0}{2} ca(\alpha, 1, \psi)$

Dijagrami: a)  $u-t$  (tanka linija) i  $x-t$  (debela linija), b)  $X-t$  i c)  $y-t$  za  $u(0)=0.3$  i  $x(0)=0.12$



Dijagrami: a)  $u-t$  (tanka linija) i  $x-t$  (debela linija), b)  $X-t$  i c)  $y-t$  za  $u(0)=0.3$  i  $x(0)=0.10$

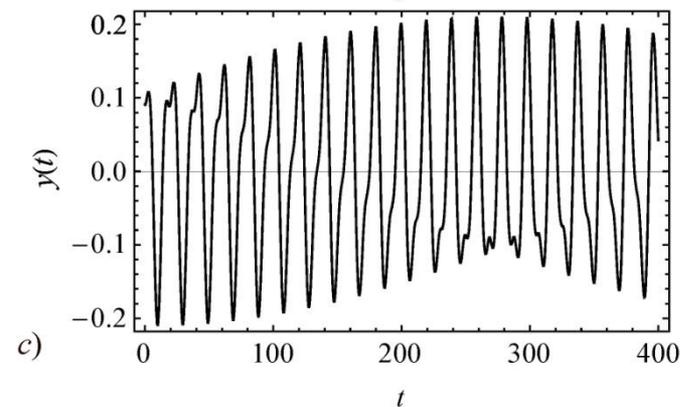
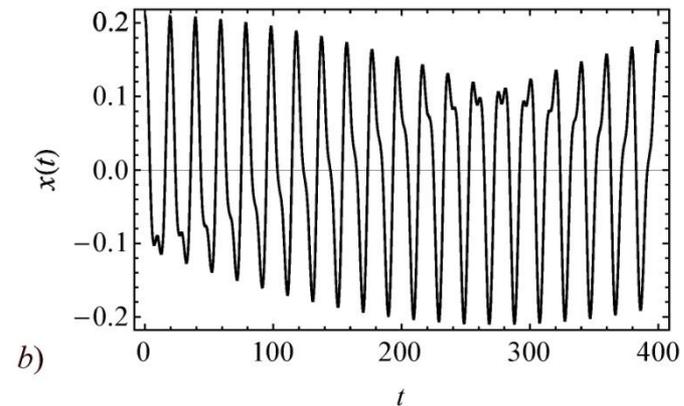
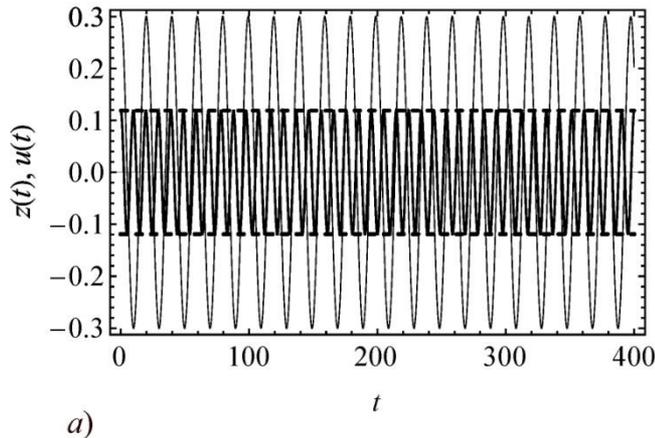


# STACIONARNE VIBRACIJE:

Dijagrami: a)  $u-t$  (tanka linija) i  $x-t$  (debela linija),

b)  $X-t$  i c)  $y-t$  za

$$u(0)=0.3 \text{ i } x(0)= 0.11894$$

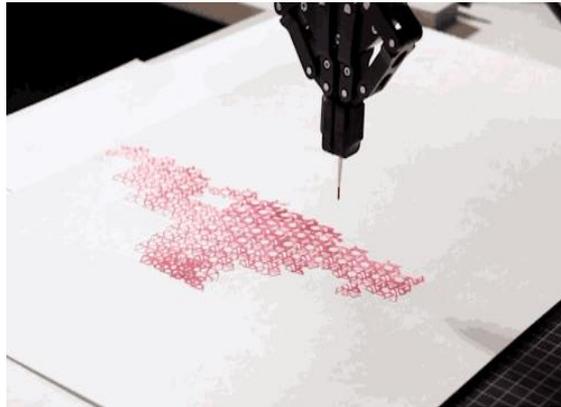


# Robotska ruka

- Primena u industriji:
  - Zavarivanje



- Farbanje



- Sečenje



# Primena u medicini

- Hirurgija

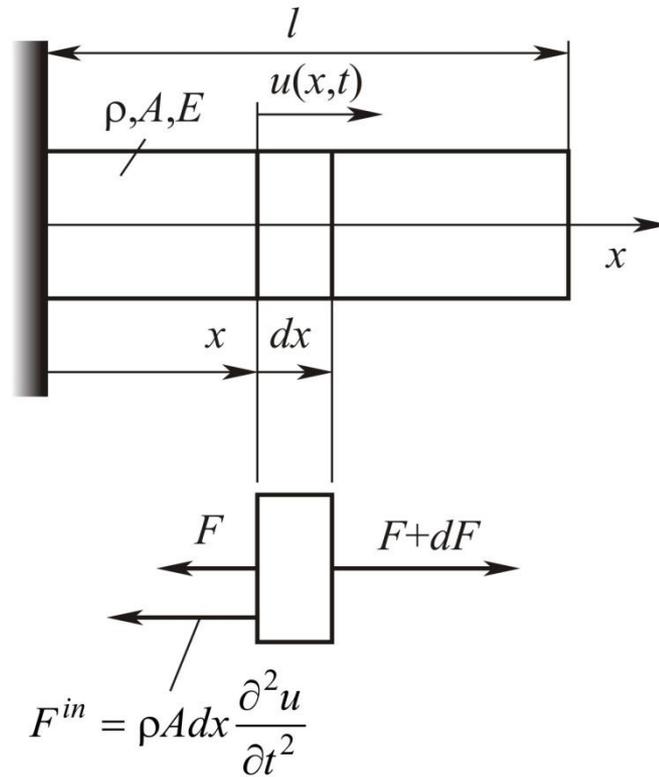


# Ruka sa iglom

- Kardiovaskularna hirurgija,
- Hirurgija jetre i pankreasa,
- Laparaskopska hirurgija



# Fizički model



# Matematički model

- $\sigma = E\varepsilon^\alpha = E(\partial u/\partial x)^\alpha$
- $F = \sigma A = EA(\partial u/\partial x)^\alpha$
  
- $\rho A \frac{\partial^2 u}{\partial t^2} = EA \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right)^\alpha$

Dve granične vrednosti:

- $u(0, t) = 0, \quad F(l, t) = EA \left( \frac{\partial u}{\partial x} \right)^\alpha (l, t) = 0,$

Dve početne vrednosti.

# Procedura za rešavanje

- $u(x, t) = X(x)T(t)$
- $\ddot{T} + c_1^2 T^\alpha = 0,$
- $\alpha(X')^{\alpha-1} X'' + k^2 X = 0.$
  
- $k^2 = \text{const.}$
- $c_1^2 = (E/\rho)k^2$
- $X(0) = 0, \quad X'(l) = 0.$

# Jednačina koja je funkcija pomeranja

- Prvi integral

- $\frac{|X'|^{\alpha+1}}{\alpha+1} + \frac{k^2 X^2}{2} = \text{const.} = K_1$

1. Koristeći granični uslov, sledi

- $X(l) = \sqrt{\frac{2K_1}{k^2}}$

# Analiza jednačine

2. Oba člana na levoj strani jednačine su nenegativna, pa je fazna trajektorija Laméova superelipsa u  $X-X'$  faznoj ravni.

Postoji jedna ravnotežna tačka  $X=X'=0$  – centar.

Sledi: **Rešenje je periodična funkcija  $x$ .**

# Približno rešenje

- $X = A \sin(Kx + \gamma_1)$ ,
- Zamena u granične uslove:
- $X(0) = A \sin \gamma_1 = 0$ ,
- $X'(l) = AK \cos(Kl + \gamma_1) = 0$ .
- Odn.
- $\gamma_1 = 0, \quad \cos(Kl) = 0$

# Karakteristična jednačina

- $K = \frac{(2n-1)\pi}{2l}, \quad n = 1, 2, 3, \dots$
- Rešenje isto kao za linearan slučaj, jer su granični uslovi linearni.
- Amplituda pozicione funkcije je:
- $A = X(l) = \sqrt{\frac{2K_1}{k^2}}$
- $k=?$

# Period funkcije od $x$

- Iz izraza za prvi integral je:

- $$X' = \frac{dX}{dx} = (\alpha + 1)^{1/(\alpha+1)} \left( K_1 - \frac{k^2}{2} X^2 \right)^{1/(\alpha+1)}$$

- odn.

- $$\int_0^{\bar{X}} dx = \frac{4}{(\alpha+1)^{1/(\alpha+1)} K_1^{1/(\alpha+1)}} \int_0^{X(l)} \frac{dX}{\left( 1 - \frac{k^2}{2K_1} X^2 \right)^{1/(\alpha+1)'}}$$

- gde je  $\bar{X}$  - period funkcije.

# Frekvencija funkcije

- $\bar{X} = \frac{4A^{\frac{\alpha-1}{\alpha+1}} B\left(\frac{\alpha}{\alpha+1}, \frac{1}{2}\right)}{(2^\alpha (\alpha+1) k^2)^{1/(\alpha+1)}}$
- Frekvencija
- $K = \frac{2\pi}{\bar{X}} = \frac{\pi (\alpha+1)^{1/(\alpha+1)} k^{2/(\alpha+1)} A^{\frac{1-\alpha}{\alpha+1}}}{2^{1/(\alpha+1)} B\left(\frac{\alpha}{\alpha+1}, \frac{1}{2}\right)}$

- Frekvencija izjednačena sa podatkom za granični uslov daje
- $k =$ 

$$\sqrt{\frac{2}{\alpha+1}} \left( \frac{(2n-1)}{2l} B \left( \frac{\alpha}{\alpha+1}, \frac{1}{2} \right) \right)^{(\alpha+1)/2} A^{(\alpha-1)/2}.$$
- Zavisi od A i reda nelinearnosti.

# Jednačina sa vremenskom funkcijom

- Tačno rešenje

- $T = Cca$

- $\left( \alpha, 1, \sqrt{\frac{E\alpha}{\rho}} \left( \frac{(2n-1)}{2l} B \left( \frac{\alpha}{\alpha+1}, \frac{1}{2} \right) \right)^{\frac{\alpha+1}{2}} (AC)^{\frac{\alpha-1}{2}} t + \right.$

- $(AC) = \text{const.}$

# Napomena

- Frekvencija oscilovanja zavisi od:
  - Početnih i graničnih uslova štapa
  - Reda i koeficijenta nelinearnosti.
- Za linearni oscilator: zavisi od graničnih uslova
- $\omega_{n1} = \sqrt{\frac{E}{\rho}} k_{n1}(K_1)$

# Približno rešenje za jednu vrednost $n$

- $u_n = D_n \sin\left(\frac{(2n-1)\pi}{2l} x\right)$

- $ca\left(\alpha, 1, \sqrt{\frac{E\alpha}{\rho}} \left(\frac{(2n-1)}{2l} B\left(\frac{\alpha}{\alpha+1}, \frac{1}{2}\right)\right)^{\frac{\alpha+1}{2}} D_n^{\frac{\alpha-1}{2}} t\right)$

gde  $D_n$  zavisi od početnih uslova

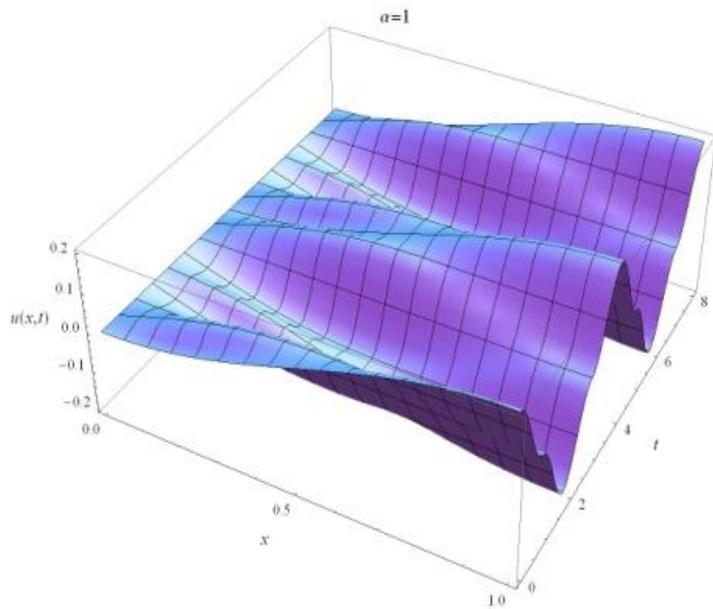
# Frekvencija oscilovanja

- Na osnovu frekvencije Ateb funkcije, frekvencija oscilovanja je

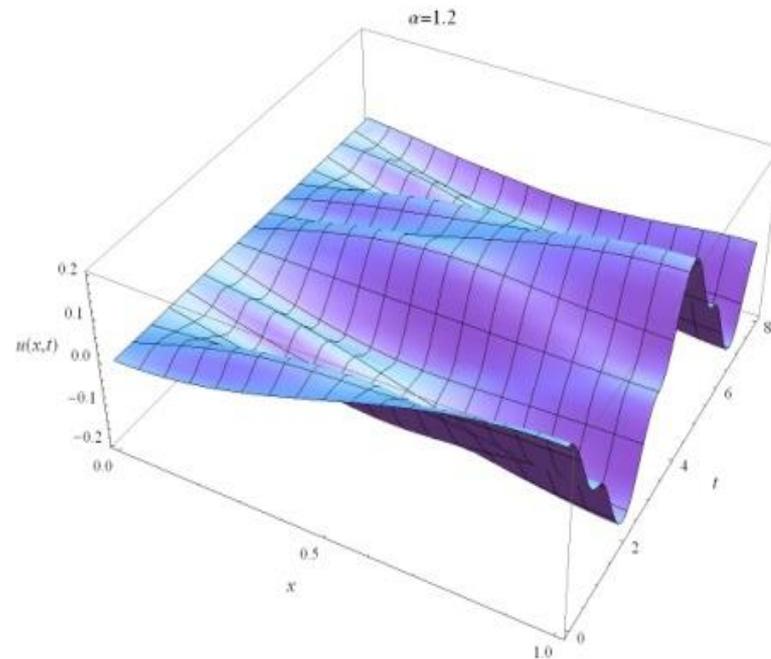
- $\omega \equiv \omega_{n,\alpha} = \frac{2\pi}{P} =$

$$\frac{\pi}{B\left(\frac{1}{\alpha+1}, \frac{1}{2}\right)} \sqrt{\frac{E\alpha}{\rho}} D_n^{\frac{\alpha-1}{2}} \left( \frac{(2n-1)}{2l} B\left(\frac{\alpha}{\alpha+1}, \frac{1}{2}\right) \right)^{\frac{\alpha+1}{2}} .$$

# Poređenje analitičkog i numeričkog rešenja

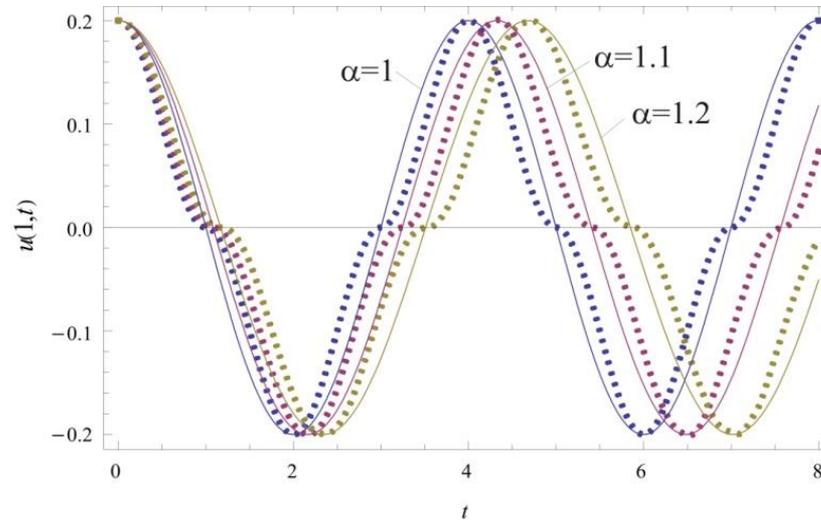


$u-x-t$  dijagram za  $\alpha=1$ .



$u-x-t$  dijagram za  $\alpha=1.2$ .

# Poređenje perioda oscilovanja



$\alpha$	$P_A$ (s)	$P_N$ (s)	Error %
1	4	4	0
1.1	4.32415	4.34737	0.537
1.2	4.68039	4.73146	1.068

### 3. Oscilator sa nelinearnim prigušenjem

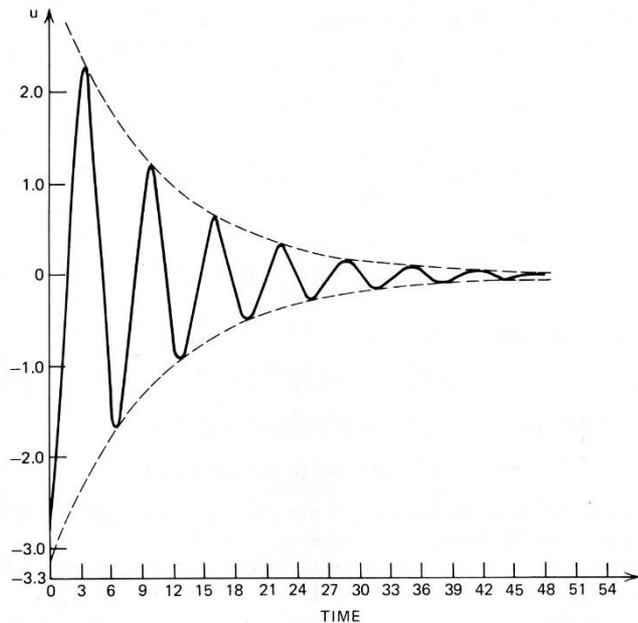
# Vrste prigušenja

- Pozitivno prigušenje
  - Suvo (Kulonovo) trenje
  - Viskozno prigušenje
  - Nelinearno prigušenje
- Negativno prigušenje

# Viskozno prigušenje

$$\ddot{x} + \omega^2 x + f\left(\frac{\dot{x}}{h}\right) = 0$$

- Kontakt tela i površine sa tankim filmom
- Telo kroz fluid sa malim Re



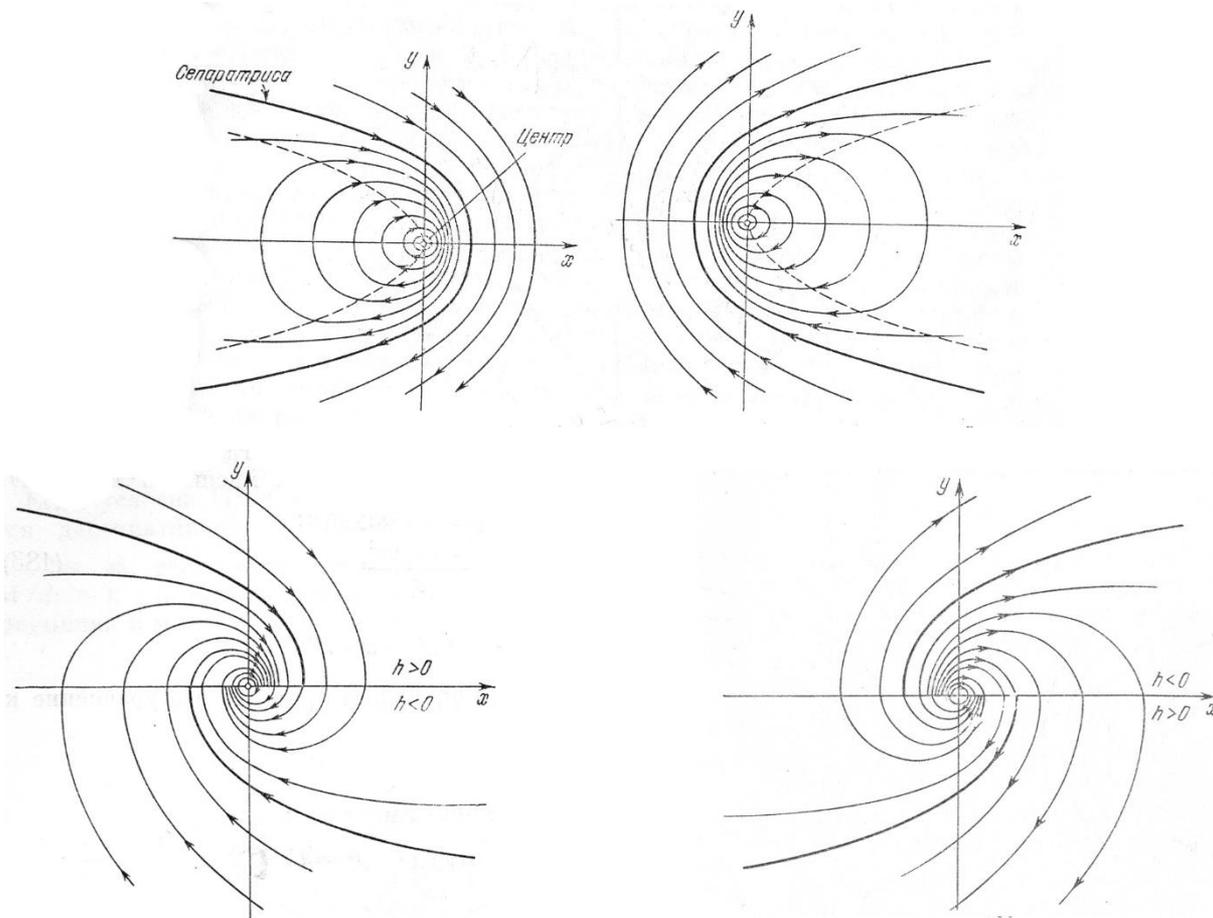
Linearna sila prigušenja

$$\ddot{u} + \omega_0^2 u = -2\epsilon\mu\dot{u}$$

$$a_0 \exp(-\epsilon\mu t) \cos(\omega_0 t + \beta_0) + O(\epsilon)$$

# Kvadratno prigušenje

$$\ddot{x} + \omega^2 x + hx^2 = 0$$



Andronov, A.A., Vitt, A.A., Hajkin, S.E., Teorija kolebanija, Nauka, Moskva, 1981.

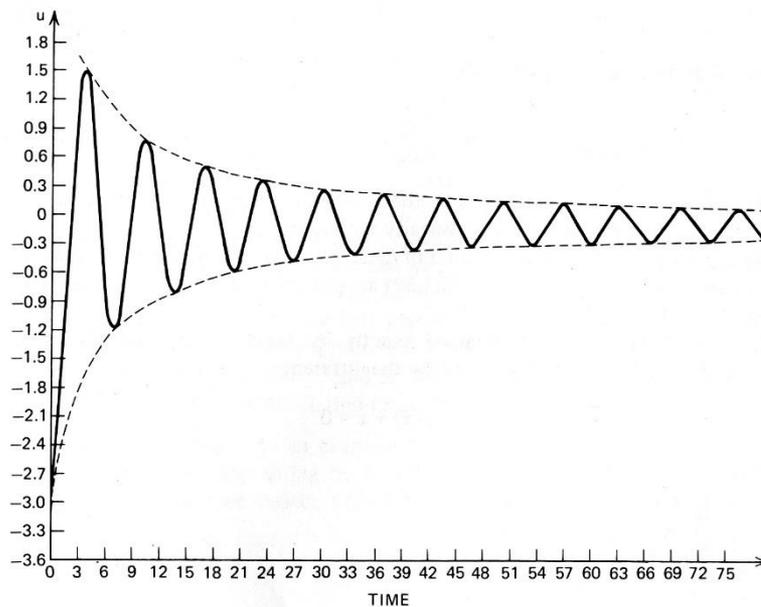
## Kvadratno prigušenje

-Telo kroz fluid sa velikim Re

$$\left. \begin{array}{l} \frac{d^2x}{dt^2} + \alpha \left( \frac{dx}{dt} \right)^2 + \omega^2 x = 0, \quad \frac{dx}{dt} > 0, \\ \frac{d^2x}{dt^2} - \alpha \left( \frac{dx}{dt} \right)^2 + \omega^2 x = 0, \quad \frac{dx}{dt} < 0, \end{array} \right\} x = a \cos \psi$$

$$a = \frac{a_0}{1 + \frac{4\alpha\omega a_0}{3\pi} t}$$

$$\psi = \omega t - \frac{3C\alpha a_0}{\pi} \left\{ 1 - \frac{1}{1 + \frac{4\alpha\omega a_0}{3\pi} t} \right\} + \psi_0$$



$$\frac{d}{dt}\left(\frac{\dot{x}^2}{2} + \frac{x^2}{2}\right) = \mp 2\delta\dot{x}^3$$

$$E = \frac{x^2}{2} + \frac{x}{(\pm 4\delta)} + \frac{1}{(\pm 4\delta)^2} + \left(\eta_i - \frac{1}{(\pm 4\delta)^2}\right) \exp((\pm 4\delta)x)$$

$$\eta_i = \frac{1}{16\delta^2} + \left(E_{i-1} - \frac{x_{i-1}^2}{2} - \frac{x_{i-1}}{(\pm 4\delta)} - \frac{1}{16\delta^2}\right) \exp(-(\pm 4\delta)x_{i-1})$$

$$x_i - \frac{1}{(\pm 4\delta)} = (\pm 4\delta)\left(\eta_i - \frac{1}{16\delta^2}\right) \exp(-(\pm 4\delta)x_i)$$

$$x_i = \frac{1}{(\pm 4\delta)} \left(1 + w\left(\frac{16\delta^2\eta_i - 1}{e}\right)\right)$$

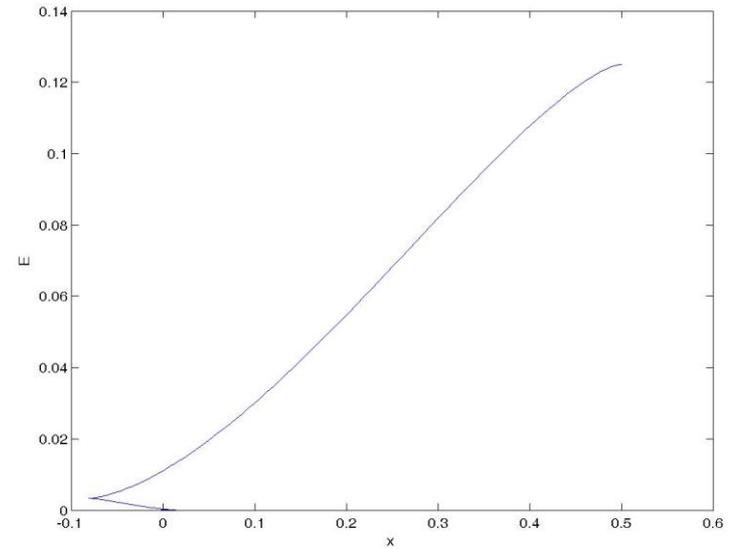
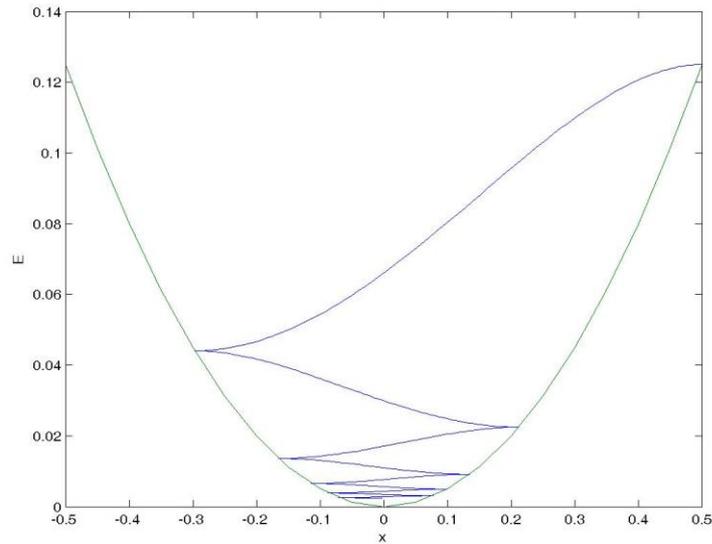
$$\begin{aligned} & w\left(-\frac{1}{16\delta^2 e} \left(\frac{x_{i-1}}{(\pm 4\delta)} + \frac{1}{16\delta^2}\right) \exp(-(\pm 4\delta)x_{i-1})\right) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{2n-1} n^{n-2}}{(n-1)!} \left(\frac{x_{i-1}}{(\pm 4\delta)^3} + \frac{1}{(16\delta^2)^2}\right)^n \exp(-n(1 + (\pm 4\delta)x_{i-1})) \end{aligned}$$

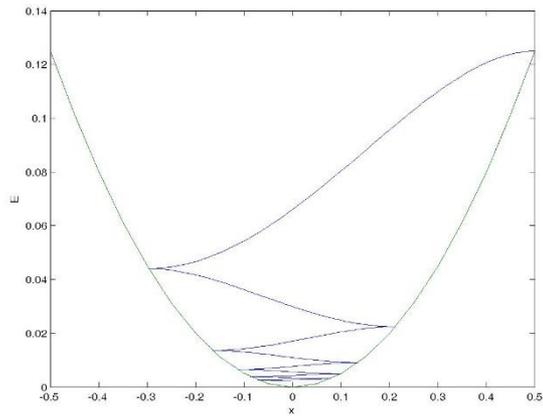
$$\ln \left| 2E + 2\delta x \sqrt{2E - x^2} \right| - \frac{2\delta}{\sqrt{1 - \delta^2}} \tan^{-1} \frac{\sqrt{2E - x^2} + \delta x}{x\sqrt{1 - \delta^2}} = \eta_i$$

$$\eta_i = \ln |2E_{i-1}| - \frac{2\delta}{\sqrt{1 - \delta^2}} \tan^{-1} \frac{\delta}{\sqrt{1 - \delta^2}}$$

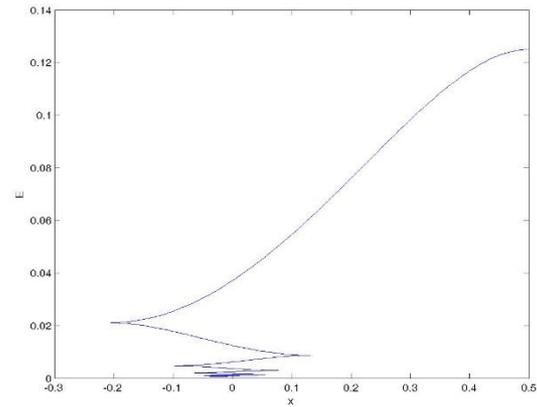
$$\ln |2E_{i-1}^0| + \frac{\delta\pi}{\sqrt{1 - \delta^2}} = \eta_i$$

# E-x dijagram za kvadratno i linearno prigušenje

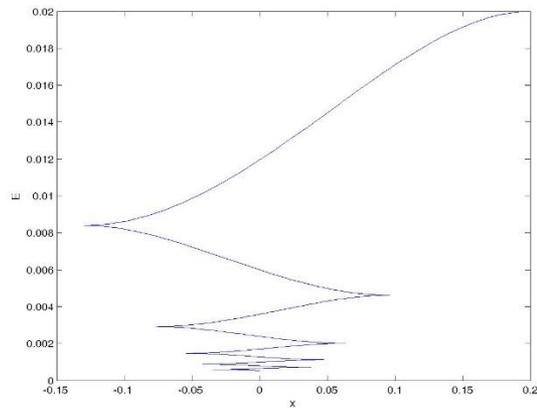




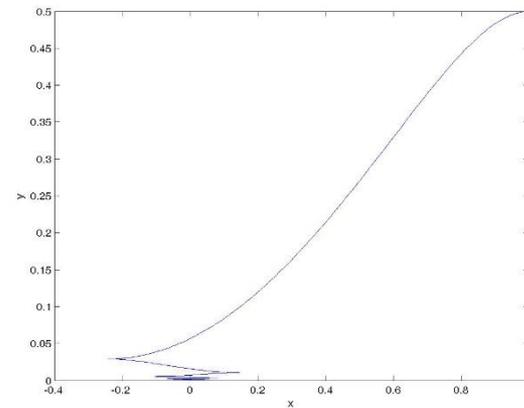
$\delta=0.5, x_0=0.5, E_0=0.125.$



$\delta=1, x_0=0.5, E_0=0.125.$



Quadratic damping  $\delta=1, x_0=0.2, E_0=0.02.$



Quadratic damping  $\delta=1, x_0=1, E_0=0.5$

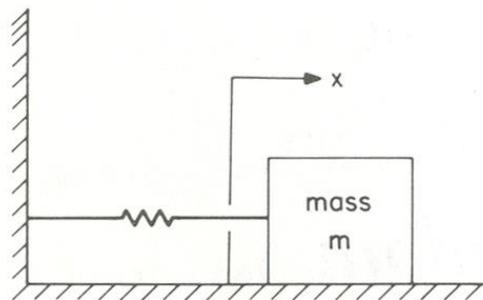
$$x = A + Bcn^2(\omega t + \alpha, k^2) \equiv A + Bcn^2$$

$$j^2 = \frac{3 - 2(\pm 4\delta)B}{2(16\delta^2 B^2 - 3(\pm 4\delta)B + 6)}, \quad k^2 = \frac{(\pm 4\delta)B}{2(\pm 4\delta)B - 3}, \quad A = \frac{(\pm 4\delta)B^2 - 3B}{16\delta^2 B^2 - 3(\pm 4\delta)B + 6}$$

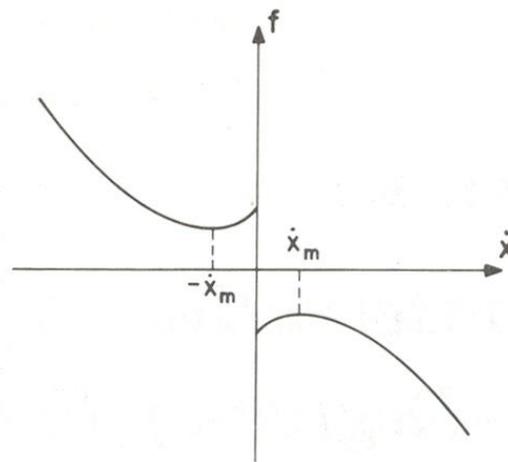
$$T_{i-1} = \sum_0^{i-1} \frac{K(k_n^2)}{\omega_n}, \quad x_{i-1} = A_{i-1}, \quad \dot{x}_{i-1} = 0$$

$x_{ie}$	0.50000	-0.296 81	0.21197	-0.16504	0.13515	-0.11448	0.09929	-0.08766
$x_{ia}$	0.50000	-0.29804	0.19916	-0.16800	0.13105	-0.11601	0.09666	-0.09010

# Suvo Kulonovo trenje



(a)



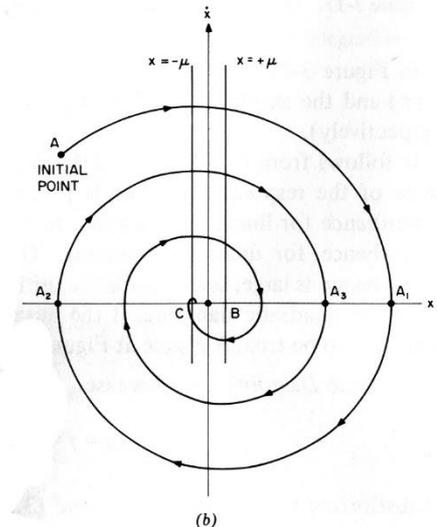
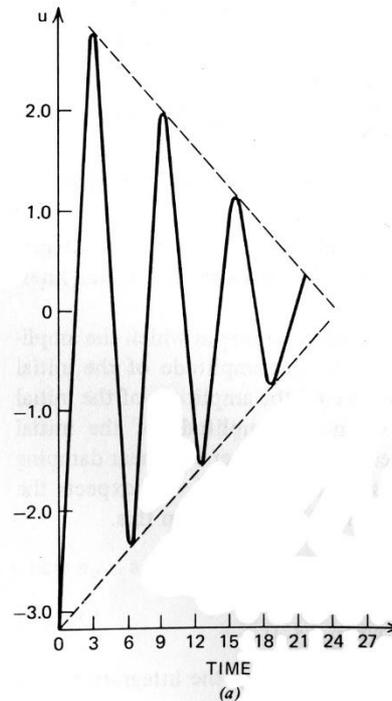
(b)

# Sila Kulonovog trenja je const.

$$\ddot{u} + \omega_0^2 u = f = \begin{cases} -\mu & \text{when } \dot{u} > 0 \\ \mu & \text{when } \dot{u} < 0 \end{cases}$$

$$a = a_0 - \frac{2\epsilon\mu}{\pi\omega_0} t, \quad \beta = \beta_0$$

$$u = a \cos(\omega_0 t + \beta_0).$$



# Suvo i viskozno trenje

$$\ddot{x} + \omega_0^2 x + 2\mu_1 \dot{x} + \mu_0 \operatorname{sgn}(\dot{x}) = 0$$

$$x = a \cos(\omega_0 t + \beta_0).$$

$$a = \left( a_0 + \frac{2\mu_0}{\pi\omega_0\mu_1} \right) \exp(-\epsilon\mu_1 t) - \frac{2\mu_0}{\pi\omega_0\mu_1}$$

$$\ddot{x} + \omega^2 x + (2\delta) \operatorname{sgn}(\dot{x}) |\dot{x}|^{m/q} = 0$$

$$x(0) = A, \quad \dot{x}(0) = 0,$$

$$\operatorname{sgn}(\dot{x}) = \begin{bmatrix} -1 & \dot{x} < 0 \\ 0 & \text{for } \dot{x} = 0 \\ 1 & \text{for } \dot{x} > 0 \end{bmatrix}$$

$$x_0 = A \exp(-\delta_0 t) (\cos(\kappa_0 t) + \frac{\delta_0}{\kappa_0} \sin(\kappa_0 t))$$

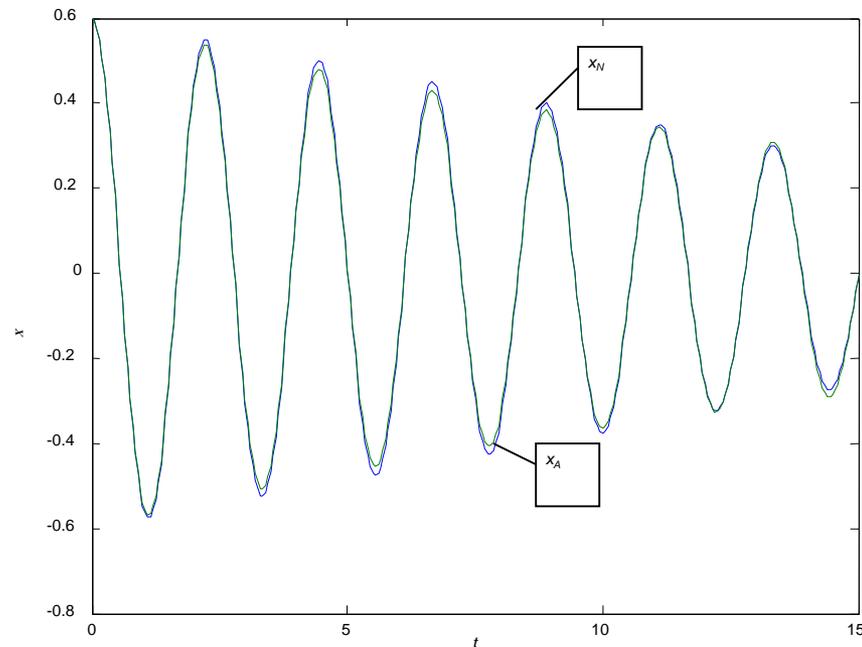
$$\omega_0^2 \approx \omega^2 [1 - 2(-1)^{n-1} a_1 (\frac{\delta}{\kappa}) (\frac{\kappa}{A\omega^2})^{1-m/q}]$$

$$\delta_0 \approx (-1)^{n-1} \frac{\delta}{\kappa} (\frac{\kappa}{A\omega^2})^{1-m/q} (\kappa b_1 - \delta a_1)$$

# Sistem sa malim koeficijentom prigušenja

$$\ddot{x} + \omega^2 x + \varepsilon \operatorname{sgn}(\dot{x}) |\dot{x}|^{m/n} = 0$$

$$x = A_0 \cos \omega t + \varepsilon A_1 \cos \omega t + (-1)^{n-1} \varepsilon (K_1 + C_2^* \cos 2\omega t)$$



$$D = (A\omega)^{m/q} \frac{2\delta a'_0}{\omega^2} \left(1 - \frac{2}{3} \frac{a'_2}{a'_0}\right)$$

$A\omega = 1.6971$

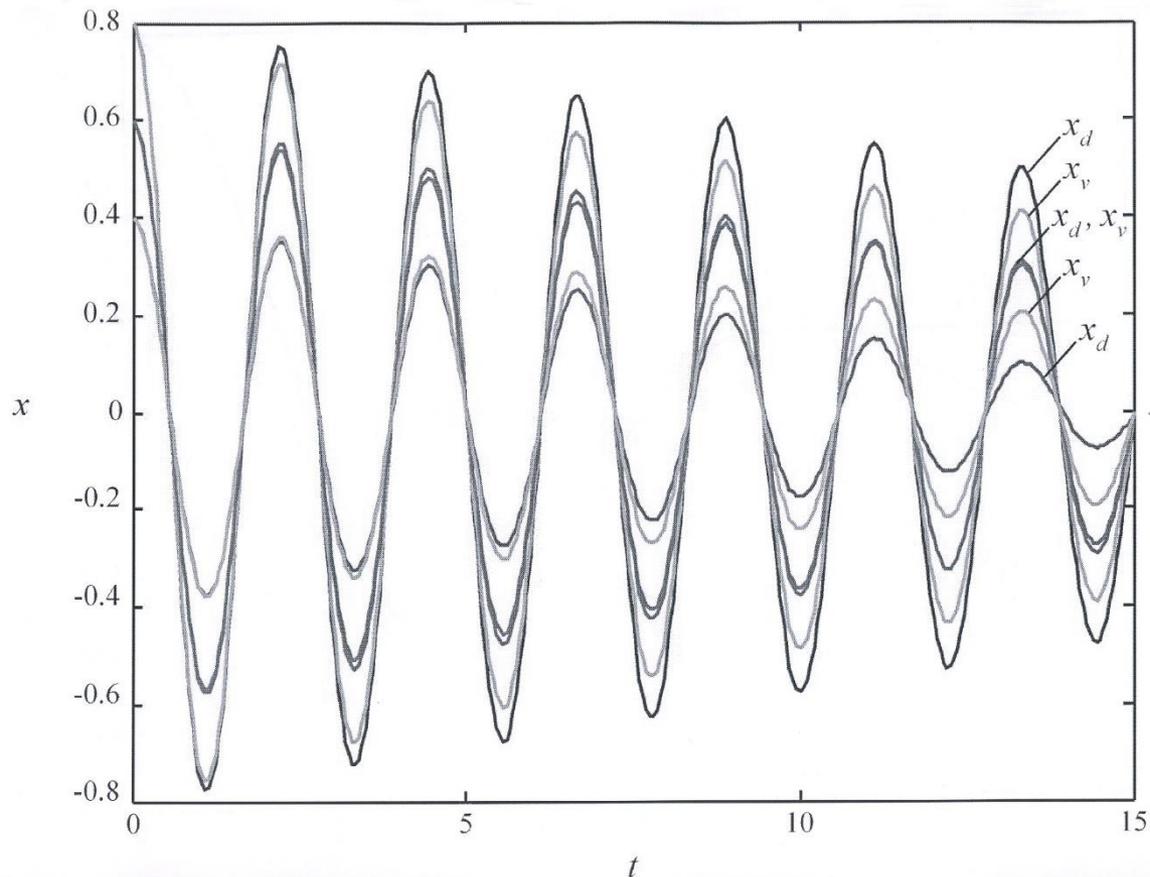
$A\omega > 1.6971$

$A\omega < 1.6971$

Dekrement opadanja amplitude je priblizno isti za viskozno prigusenje ( $m/n=1$ ) I suvo trenje ( $m/n=0$ ).

Amplituda opada brze kod viskoznog prigusenja nego kod suvog trenja.

Prigusenje je brze kod subog trenja nego viskoznog.

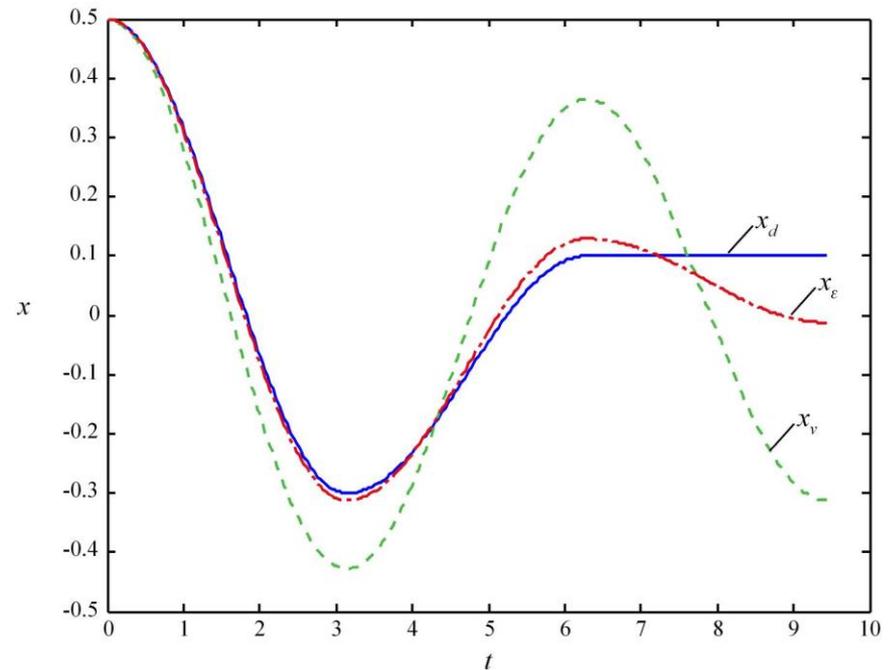
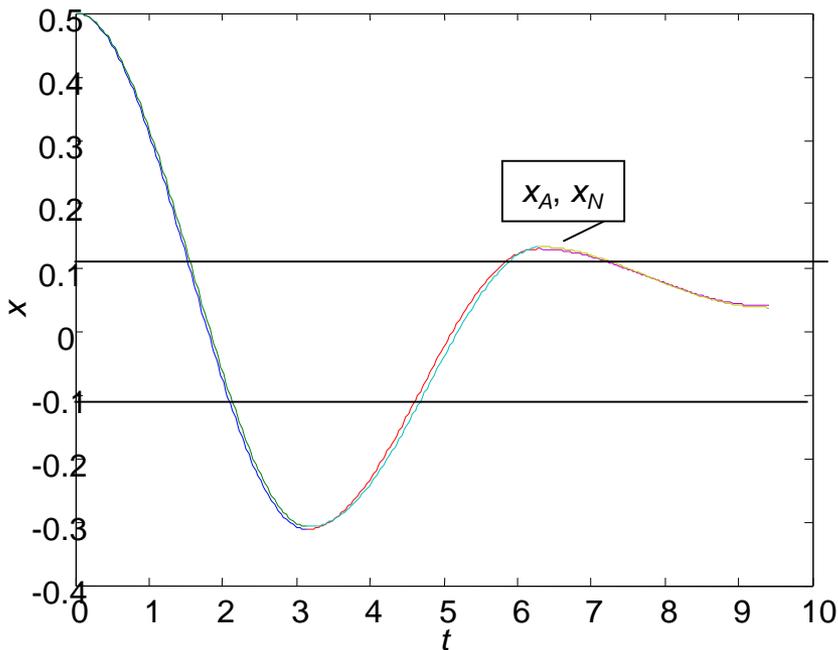


# Sistem sa prigušenjem bliskim konstantnom suvom trenju

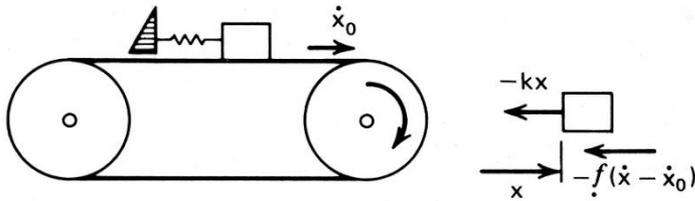
$$\ddot{x} + \omega^2 x = (2\delta) |\dot{x}|^\varepsilon$$

$$\ddot{x} + \omega^2 x = -(2\delta) |\dot{x}|^\varepsilon$$

$$x_n = (-1)^{n-1} \left\{ \frac{2\delta}{\omega^2} + B_{n-1} \left( 1 + \frac{2\delta P(\varepsilon)t}{(B_{n-1}\omega)^{1-\varepsilon}} \right) \sin \left[ \left( \omega - (-1)^{n-1} \frac{(2\delta)Q(\varepsilon)}{(B_{n-1}\omega)^{1-\varepsilon}} \right) t + \alpha_{n-1} \right] \right\}$$



# Samopobudni mehanički oscilator



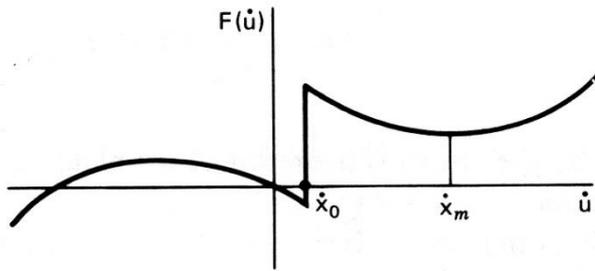
(a)

$$m\ddot{x} + kx - f(\dot{x} - \dot{x}_0) = 0$$

$$u = x - k^{-1} f(-\dot{x}_0)$$

$$\ddot{u} + \omega_0^2 u + F(\dot{u}) = 0$$

$$F(\dot{u}) = m^{-1} [f(-\dot{x}_0) - f(\dot{u} - \dot{x}_0)]$$

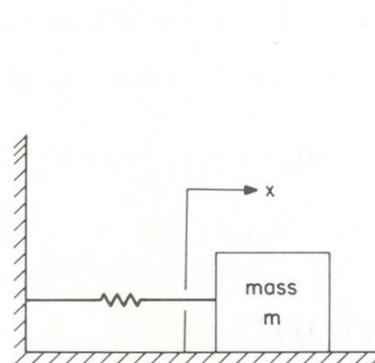


(b)

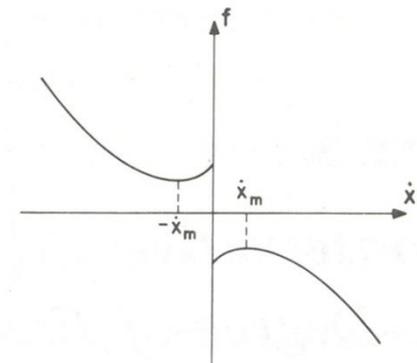
$$\ddot{u} + \omega_0^2 u - \varepsilon(c_0 \dot{x} - c_2 \dot{x}^3) = 0$$

Rejlijeva jednačina

Van der Polova jednačina

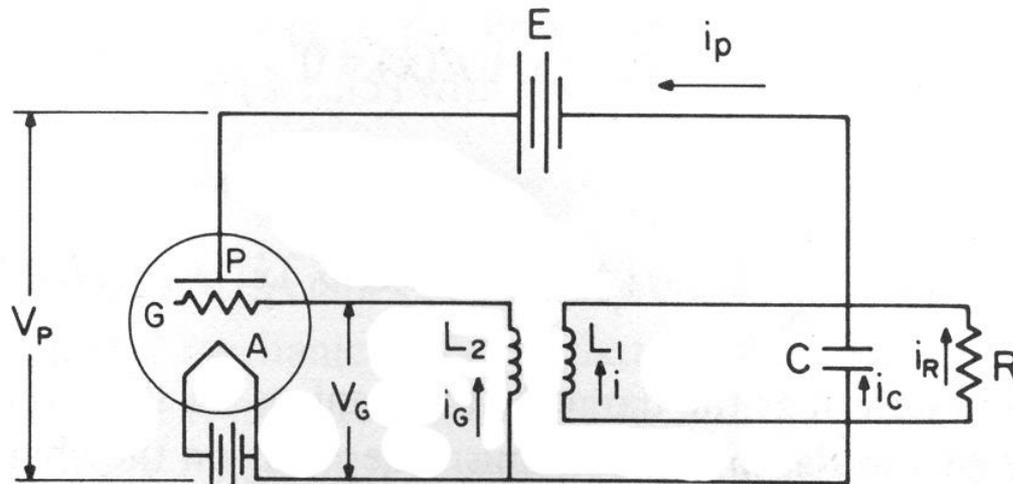


(a)



(b)

# Van der Polovo električno kolo



$$L_1 C \frac{d^2 i}{dt^2} + \frac{L_1}{R} \frac{di}{dt} + i = \phi \left[ \Delta E + (M - \Delta L_1) \frac{di}{dt} \right]$$

# Rejlijeva jednačina (van der Polova jednačina)

$$x = i - \phi(\Delta E)$$

$$L_1 C \ddot{x} + x + F(\dot{x}) = 0$$

$$F(\dot{x}) = \frac{L_1}{R} \dot{x} - \phi[\Delta E + (M - \Delta L_1) \dot{x}] + \phi(\Delta E)$$

$$F(\dot{x}) = -\alpha_1 \dot{x} + \alpha_3 \dot{x}^3$$

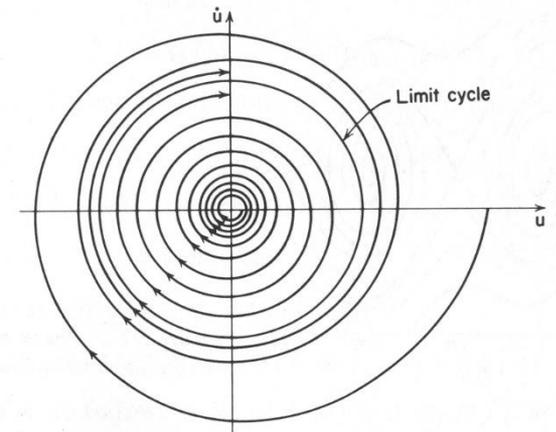
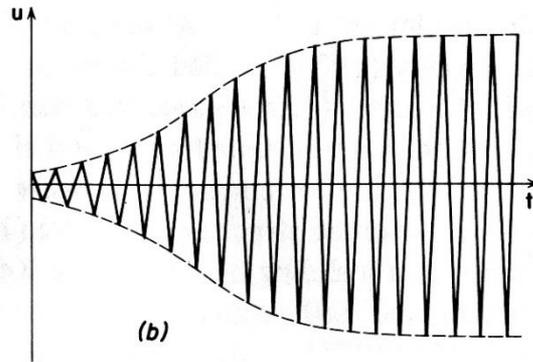
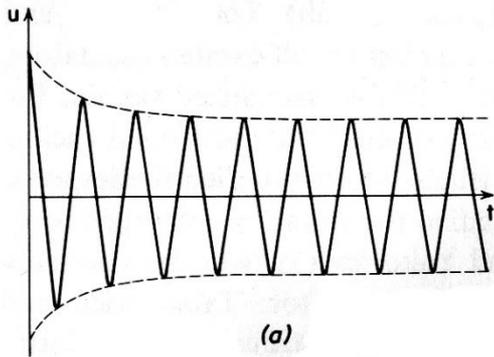
$$x = u \sqrt{\alpha_1 / \alpha_3}$$

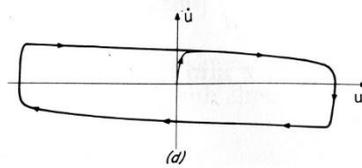
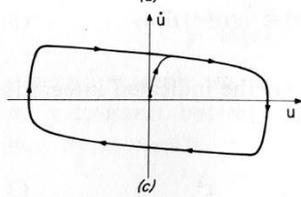
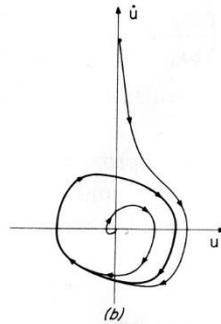
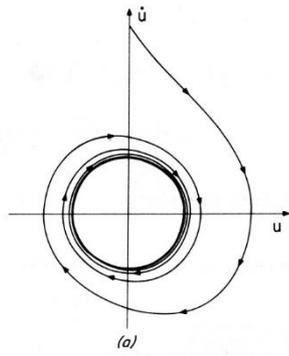
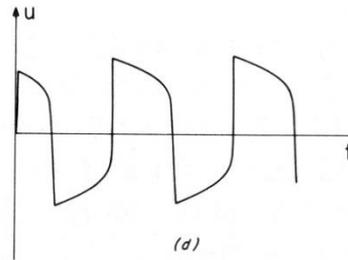
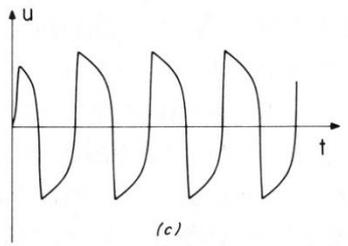
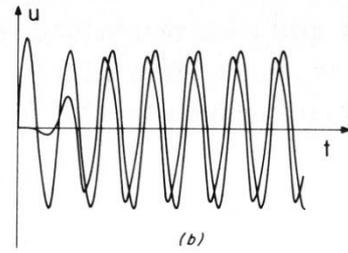
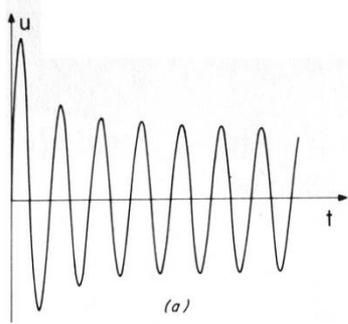
$$\ddot{u} + \omega_0^2 u - \epsilon(\dot{u} - \dot{u}^3) = 0$$

$$\ddot{v} + \omega_0^2 v - \epsilon(1 - v^2)\dot{v} = 0, \quad v = \sqrt{3}\dot{u}$$

$$\ddot{u} + \omega_0^2 u = \epsilon(\dot{u} - \dot{u}^3)$$

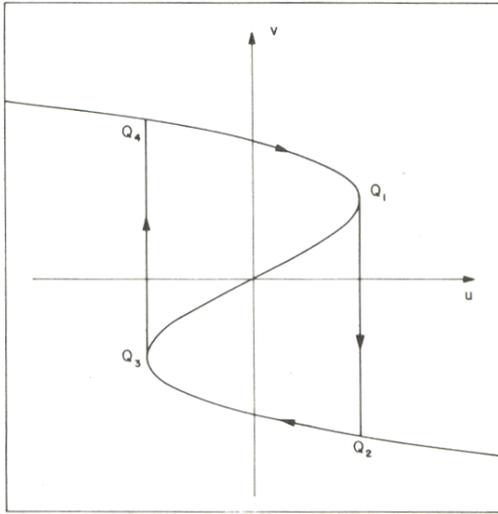
$$a^2 = \frac{a_0^2}{\frac{3}{4} \omega_0^2 a_0^2 + (1 - \frac{3}{4} \omega_0^2 a_0^2) \exp(-\epsilon t)}$$





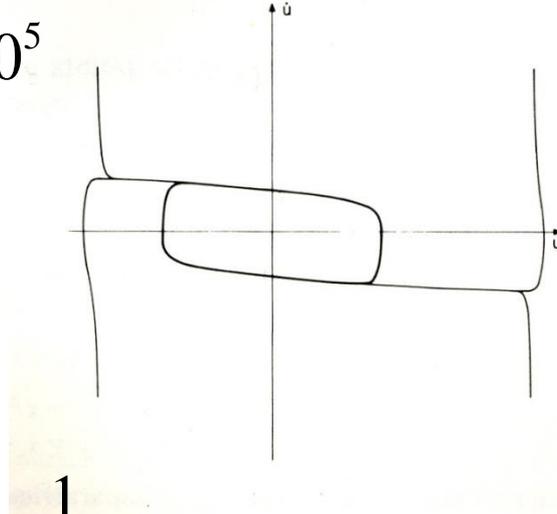
# Relaksacione oscilacije

$$\ddot{u} + \omega_0^2 u = \epsilon(\dot{u} - \dot{u}^3)$$



$$\dot{u} = v \quad \xi = \frac{1}{\epsilon} u$$

$$\epsilon = 10^5$$



$$\dot{\xi} = \frac{1}{\epsilon} v$$

$$\frac{1}{\epsilon} \dot{v} = \xi + v - v^3$$

$$\epsilon^2 \frac{dv}{d\xi} = \frac{\xi + v - v^3}{v}$$

Van der Pol, 1922.

$$\ddot{x} + c_1 x + c_3 x^3 - \varepsilon (c_0 \dot{x} - c_2 \dot{x}^3) = 0$$

$$c_1 > 0 \quad c_3 > 0$$

$$x_0 = a \operatorname{cn}(\omega t, k^2) \equiv a \operatorname{cn}$$

$$\frac{1}{3}[(1-k^2) - (1-2k^2)] \frac{E}{K} = \left(\frac{c_1^2 c_2}{c_3 c_0}\right) \frac{1}{35k^2} [(8k^6 - 13k^4 + 3k^2 + 2) - (16k^6 - 24k^4 + 4k^2 + 2) \frac{E}{K}]$$

$$a = \sqrt{\frac{c_1}{c_3} \frac{2k^2}{(1-2k^2)}}, \quad \omega = \sqrt{\frac{c_1}{1-2k^2}}$$

Linearna elastičnost

$$x = 1.1547 \sqrt{\frac{1}{c_1 c_2} \frac{c_0}{c_1}} \cos(t \sqrt{c_1})$$

$$a = \frac{2}{\omega \sqrt{3}} \sqrt{\frac{c_0}{c_2}}$$

Čisto kubna nelinearnost

$$x = 1.2359 \sqrt[4]{\frac{1}{c_3 c_2} \frac{c_0}{c_2}} \operatorname{cn}(1.2359 t \sqrt[4]{c_3 \frac{c_0}{c_2}}, 0.5)$$

$$T = \frac{4K(0.5)}{1.2359 \sqrt[4]{c_3 \frac{c_0}{c_2}}} = 6 \sqrt[4]{\frac{1}{c_3 c_0} \frac{c_2}{c_0}}$$

$$c_1 > 0 \quad c_3 < 0$$

$$x_0 = a \operatorname{sn}(\omega t, k^2)$$

$$\frac{c_1^2}{(-c_3)} \frac{c_2}{c_0} = -\frac{35k^2}{6} \frac{[(1-k^2)K - (1+k^2)E](1+k^2)^2}{[(2-11k^2+8k^4+k^6)K - (2-10k^2-10k^4+2k^6)E]}$$

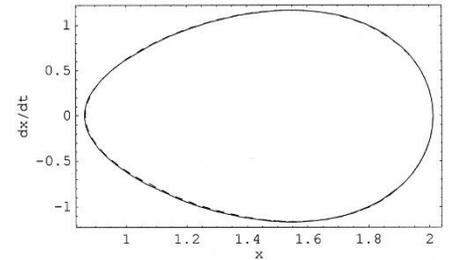
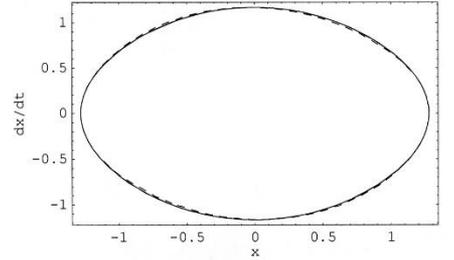
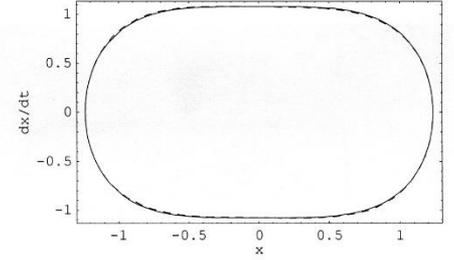
$$a = \sqrt{\frac{2c_1k^2}{(-c_3)(1+k^2)}}, \quad \omega = \sqrt{\frac{c_1}{1+k^2}}$$

$$c_1 < 0 \quad c_3 > 0$$

$$x_0 = a \operatorname{dn}(\omega t, k^2)$$

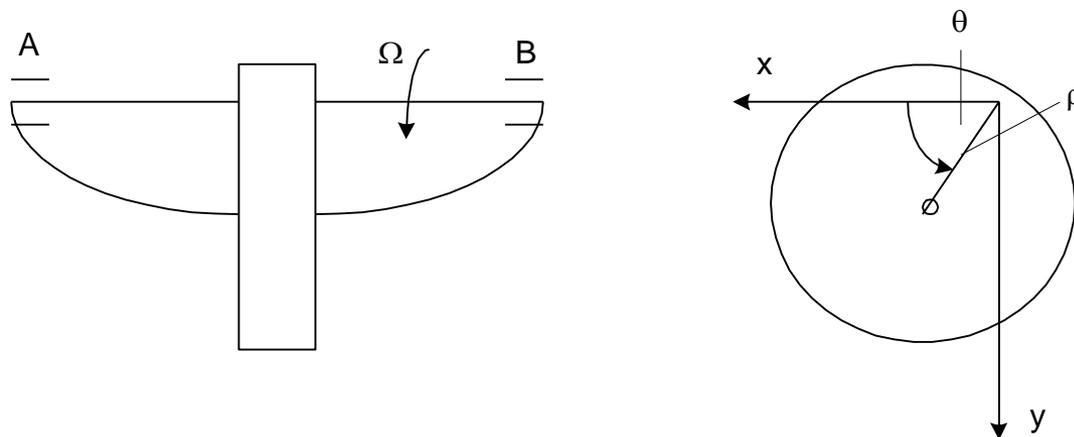
$$\frac{c_2}{c_0} \frac{(-c_1)^2}{c_3} = -\frac{35}{6} \frac{[2(1-k^2)K - (2-k^2)E](2-k^2)^2}{K(k^6+15k^4-32k^2+16) - E(2k^6+4k^4-24k^2+16)}$$

$$a = \sqrt{\frac{(-c_1)}{c_3} \frac{2}{2-k^2}}, \quad \omega = \sqrt{\frac{(-c_1)}{2-k^2}}$$



# **4. Analitičke metode određivanja oscilacija rotora**

# Model vratilo-disk sistema



$$\ddot{z} + zF(z\bar{z}) = \varepsilon\phi_z(z, \dot{z}, cc), \quad (1)$$

$$z(0) = z_0, \quad \dot{z}(0) = \dot{z}_0. \quad (2)$$

$$z = x + iy, \quad i = \sqrt{-1}.$$

# Generalizacija metode Krilov-Bogoljubov

$$\varepsilon = 0 \quad \ddot{z} + zF(z\bar{z}) = 0, \quad (3)$$

$$z = f(K_1, K_2, K_3, K_4, t). \quad (4)$$

$$\varepsilon \neq 0,$$

$$z = f(K_1(t), K_2(t), K_3(t), K_4(t), t). \quad (5)$$

$$\dot{z} = \partial f / \partial t, \quad (6)$$

$$\sum_{j=1}^4 \frac{\partial f}{\partial K_j} \dot{K}_j = 0, \quad (7)$$

$$\sum_{j=1}^4 \frac{\partial}{\partial K_j} \left( \frac{\partial f}{\partial t} \right) \dot{K}_j = \varepsilon \phi_z \left( f, \frac{\partial f}{\partial t}, cc \right). \quad (8)$$

Odnosno

$$\dot{K}_j = \frac{\Delta_{Kj}}{\Delta}, \quad j = 1, 2, 3, 4. \quad (9)$$

$$\Delta = |a_{kj}|,$$

$$a_{1j} = \operatorname{Re}(\partial f / \partial K_j), \quad a_{2j} = \operatorname{Im}(\partial f / \partial K_j),$$

$$a_{3j} = \operatorname{Re}\left(\frac{\partial}{\partial K_j}\left(\frac{\partial f}{\partial t}\right)\right), \quad a_{4j} = \operatorname{Im}\left(\frac{\partial}{\partial K_j}\left(\frac{\partial f}{\partial t}\right)\right),$$

$$\Delta_{Kj} = |b_{kp}|, \quad p = 1, 2, 3, 4.$$

$$b_{kp} = a_{kj}, \quad p = j.$$

$$j = \operatorname{index}(K_j)$$

$$b_{3j} = \varepsilon \operatorname{Re}(\phi) \quad b_{4j} = \varepsilon \operatorname{Im}(\phi)$$

$$\dot{K}_j = \left\langle \frac{\Delta_{Kj}}{\Delta} \right\rangle, (10) \quad \langle \cdot \rangle \equiv \frac{1}{T} \int_0^T (\cdot) dt,$$

$$j = 1, 2, 3, 4.$$

- Zamenom približnih rešenja  $K_j(t)$  u (5) dobija se približno rešenje diferencijalne jednačine (1).
- Proces rešavanja osrednjenih jednačina praćen je nizom poteškoća.
- Proces osrednjavanja ponekad unosi vrlo veliku grešku, pa približno rešenje važi samo za izuzetno male početne vrednosti i kratak vremenski interval.

# Unutrašnje prigušenje

$$\ddot{z} + 2\dot{z} + z(z\bar{z}) = -\varepsilon b i z,$$

$$z(0) = z_0, \quad \dot{z}(0) = 0.$$

$$\varepsilon = 0 \quad \Rightarrow \quad z = A \exp(i\theta) \operatorname{cn}(\omega t, m).$$

gde je:  $A \exp(i\theta) = z_0,$

$$m = \frac{A^2}{2(A^2 + 2)}, \quad \omega = \sqrt{2 + A^2}.$$

$$\varepsilon \neq 0$$

$$z = A(t) \exp(i\theta(t)) \operatorname{cn}(\psi(t), m(t)),$$

$$\psi = \int \omega(t) dt,$$

$$\omega(t) = \sqrt{2 + A(t)^2}, \quad m(t) = \frac{A(t)^2}{2(A(t)^2 + 2)},$$

Eliptički Krilov Bogoljubov metod

$$2\dot{A}\omega + A\dot{\omega} + A\omega\dot{m}(ns \operatorname{sn}_m + nd \operatorname{dn}_m)nc = 0,$$

$$2\omega\dot{\theta} \operatorname{sn}^2 \operatorname{dn}^2 = -\varepsilon b \operatorname{sn} \operatorname{cn} \operatorname{dn}.$$

$$A = \text{const.}, \quad \theta = \text{const.} = \theta_0. \quad z = z_0 \operatorname{cn}(\omega t, m).$$

Runge Kutta numerički metod

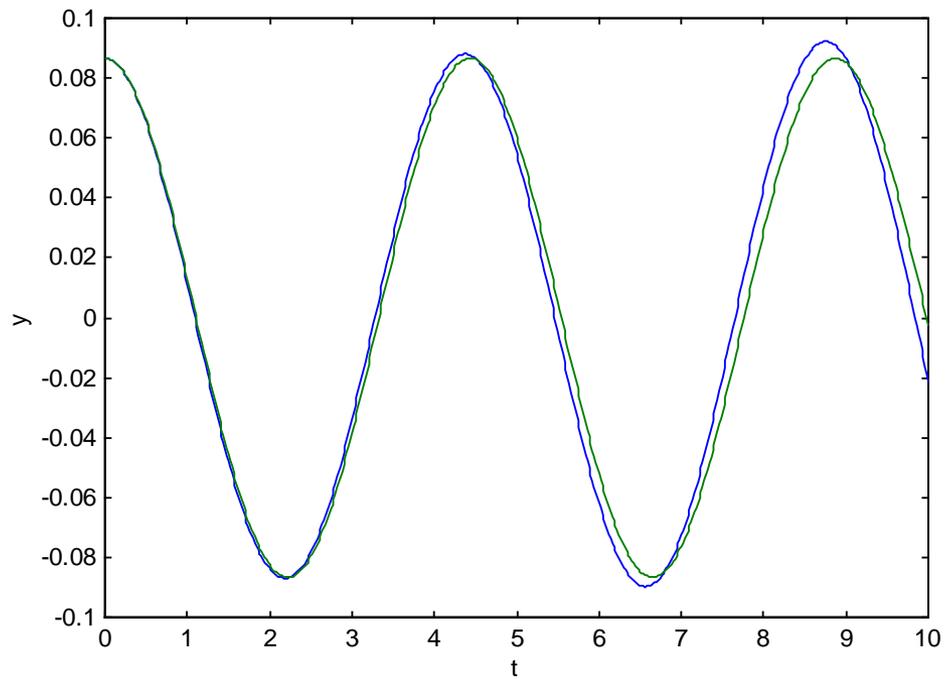
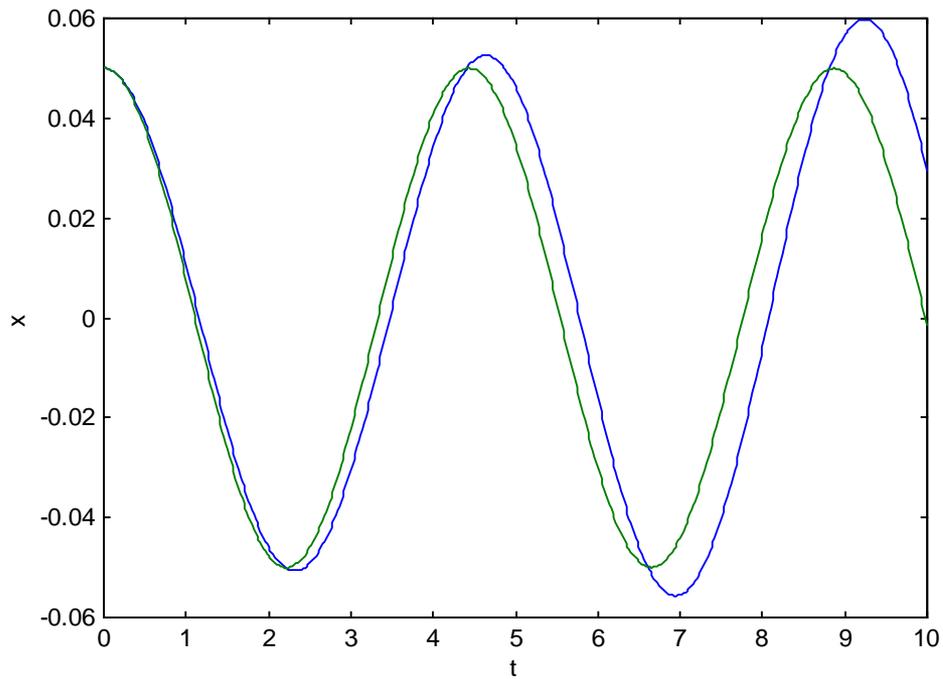
$$\ddot{x} + 2x + x(x^2 + y^2) = \varepsilon b y,$$

$$\ddot{y} + 2y + y(x^2 + y^2) = -\varepsilon b x.$$

$$x(0) = 0.05, \quad \dot{x}(0) = 0,$$

$$y(0) = 0.0866, \quad \dot{y}(0) = 0.$$

$$\varepsilon b = 0.1$$



# Diferencijalne jednačine oscilovanja

$$\begin{aligned}\ddot{\rho} - \rho\dot{\theta}^2 + \rho f(\rho) &= \varepsilon F_\rho(\rho, \dot{\rho}, \theta, \dot{\theta}), \\ \rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} &= \varepsilon F_\theta(\rho, \dot{\rho}, \theta, \dot{\theta}),\end{aligned}\tag{11}$$

## Početni uslovi

$$\begin{aligned}\rho(0) &= \rho_0, & \dot{\rho}(0) &= \dot{\rho}_0, \\ \theta(0) &= \theta_0, & \dot{\theta}(0) &= \dot{\theta}_0.\end{aligned}\tag{12}$$

$$\varepsilon = 0$$

$$\ddot{\rho} - \rho\dot{\theta}^2 + \rho f(\rho) = 0,$$

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = 0,$$

odn.

$$\rho^2\dot{\theta} = \rho_0^2\dot{\theta}_0 = K_{10} = \text{const.}, \quad (13)$$

$$\frac{1}{2}\dot{\rho}^2 + \frac{1}{2}\frac{K_{10}^2}{\rho^2} + \int \rho f(\rho) d\rho =$$

$$\frac{1}{2}\dot{\rho}_0^2 + \frac{1}{2}\frac{K_{10}^2}{\rho_0^2} + \int \rho f(\rho) d\rho \Big|_{\varepsilon=0} = K_{20} = \text{const.}, \quad (14)$$

$$\dot{\rho} = \left(2K_{20} - \frac{K_{10}^2}{\rho^2} - 2\int \rho f(\rho) d\rho\right)^{1/2}, \quad (15)$$

$$\dot{\theta} = \frac{K_{10}}{\rho^2}. \quad (16)$$

$$\rho(0) = \rho_0, \quad \theta(0) = \theta_0.$$

$$\rho_{\varepsilon 0} = \rho_{\varepsilon 0}(t), \quad \theta_{\varepsilon 0} = \theta_{\varepsilon 0}(t).$$

$$\varepsilon \neq 0$$

$$\frac{dK_1(t)}{dt} = \varepsilon \phi_1(K_1, K_2, \rho, \theta, t),$$

$$\frac{dK_2(t)}{dt} = \varepsilon \phi_2(K_1, K_2, \rho, \theta, t),$$

$$\frac{d\rho}{dt} = \left(2K_2 - \frac{K_1^2}{\rho^2} - 2 \int \rho f(\rho) d\rho\right)^{1/2},$$

$$\frac{d\theta}{dt} = \frac{K_1}{\rho^2},$$

Početni uslovi:

$$\rho(0) = \rho_0, \quad \theta(0) = \theta_0, \quad K_1(0) = K_{10}, \quad K_2(0) = K_{20},$$

gde je:

$$\varepsilon\phi_1(K_1, K_2, \rho, \theta, t) = \varepsilon\rho F_\theta(\rho, \dot{\rho}, \theta, \dot{\theta}),$$

$$\varepsilon\phi_2(K_1, K_2, \rho, \theta, t) = \varepsilon\dot{\rho} F_\rho(\rho, \dot{\rho}, \theta, \dot{\theta}),$$

$$\frac{dK_1(t)}{dt} = \varepsilon\phi_1(K_1, K_2, \rho, \theta, t) \Big|_{\varepsilon=0},$$

$$\frac{dK_2(t)}{dt} = \varepsilon\phi_2(K_1, K_2, \rho, \theta, t) \Big|_{\varepsilon=0},$$

(17)

pa je:

$$K_1 = K_{10} + \varepsilon K_{11},$$

$$K_2 = K_{20} + \varepsilon K_{21},$$

$$\rho = \rho_{\varepsilon 0} + \varepsilon \rho_1,$$

$$\theta = \theta_{\varepsilon 0} + \varepsilon \theta_1,$$

$$\frac{d\rho_1}{dt} = \frac{1}{\rho_{\varepsilon 0} \dot{\rho}_{\varepsilon 0}} \left( K_{21} \rho_{\varepsilon 0} - \frac{K_{10} K_{11}}{\rho_{\varepsilon 0}} + \rho (2K_{20} - \dot{\rho}_{\varepsilon 0}^2 - 2\phi_0) - \rho_{\varepsilon 0} \phi_1 \right), \quad (18)$$

Bernulijeva jednačina

$$\frac{d\theta_1}{dt} = \frac{K_{11}}{\rho_{\varepsilon 0}^2} - 2\rho_1 \rho_{\varepsilon 0} \dot{\theta}_{\varepsilon 0}, \quad (19)$$

$$z = (\rho_{\varepsilon 0} + \varepsilon \rho_1) \exp(\theta_{\varepsilon 0} + \varepsilon \theta_1).$$

# Bezudarni početni uslovi

$$\dot{\rho}(0) = 0, \quad \dot{\theta}(0) = 0,$$

$$K_{10} = 0, \quad \dot{\rho} = 2(K_2 - \int \rho f(\rho) d\rho)^{1/2}.$$

$\varepsilon \neq 0$

$$\rho = \rho_{\varepsilon 0}, \quad \theta = \theta_0 + \varepsilon \int_0^t \frac{K_{11}}{\rho_{\varepsilon 0}^2} dt,$$

$$z = \rho_{\varepsilon 0} \exp(\theta_0 + \varepsilon \theta_1).$$

# GIROSKOPSKA SILA

$$\ddot{\rho} - \rho\dot{\theta}^2 + 2\rho + \rho^3 = -2\varepsilon g \dot{\theta},$$

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = 2\varepsilon g \dot{\rho}.$$

$$\rho_0 = 0.1, \quad \theta_0 = \pi/3, \quad \dot{\rho}_0 = 0, \quad \dot{\theta}_0 = 0.$$

$$\varepsilon = 0$$

$$\rho = A \operatorname{cn}(\omega t, m), \quad \dot{\rho} = -A \omega \operatorname{sn}(\omega t, m) \operatorname{dn}(\omega t, m)$$

$$\theta = \theta_0, \quad \dot{\theta} = 0, \quad A = \rho_0,$$

$$\omega = \sqrt{2 + A^2}, \quad m = \frac{A^2}{2(A^2 + 2)}, \quad K_{20} = A^2 + \frac{A^4}{4}.$$

## Metod prvih integrala

$$\varepsilon \neq 0 \quad \frac{dK_2(t)}{dt} = -2\varepsilon g(\rho\dot{\rho}\dot{\theta})\Big|_{\varepsilon=0},$$

$$K_2 = K_{20} = \text{const.}, \Rightarrow A = \rho(0) = \text{const.} = \rho_0 = 0.1$$

$$m = 0.00248, \quad \omega = 1.41774,$$

$$\rho = \rho_{\varepsilon 0} = 0.1 \text{cn}(1.41774t, 0.00248).$$

Giroskopska sila ne utiče na amplitudu oscilovanja.

$$\frac{dK_1(t)}{dt} = 2\varepsilon g (\rho \dot{\rho})|_{\varepsilon=0}, \quad K_1(t) = -\varepsilon g A^2 \operatorname{sn}^2(\omega t, m).$$

$$\theta = -\varepsilon g \int_0^t \frac{\operatorname{sn}^2(\omega t, m)}{\operatorname{cn}^2(\omega t, m)} dt + \theta_0.$$

$$\theta = \frac{\varepsilon g}{1-m} [E(\omega t, m) - \operatorname{dn}(\omega t, m) \operatorname{tn}(\omega t, m)] + \theta_0.$$

$$m \approx 0, \quad \theta \approx \varepsilon g t + \theta_0.$$

$$z = \rho_0 \exp(i(\theta_0 + \varepsilon g t)) \operatorname{cn}(\omega t, m).$$

$$z = 0.1 \exp(i(\pi / 3 + 0.1t)) \operatorname{cn}(1.41774t, 0.00248).$$

# GIROSKOPSKA SILA

$$\ddot{z} + 2z + z(z\bar{z}) = 2\varepsilon g i \dot{z},$$

$$\varepsilon g = 0.1, \quad z(0) = 0.1 \exp(i\pi / 3), \quad \dot{z}(0) = 0.$$

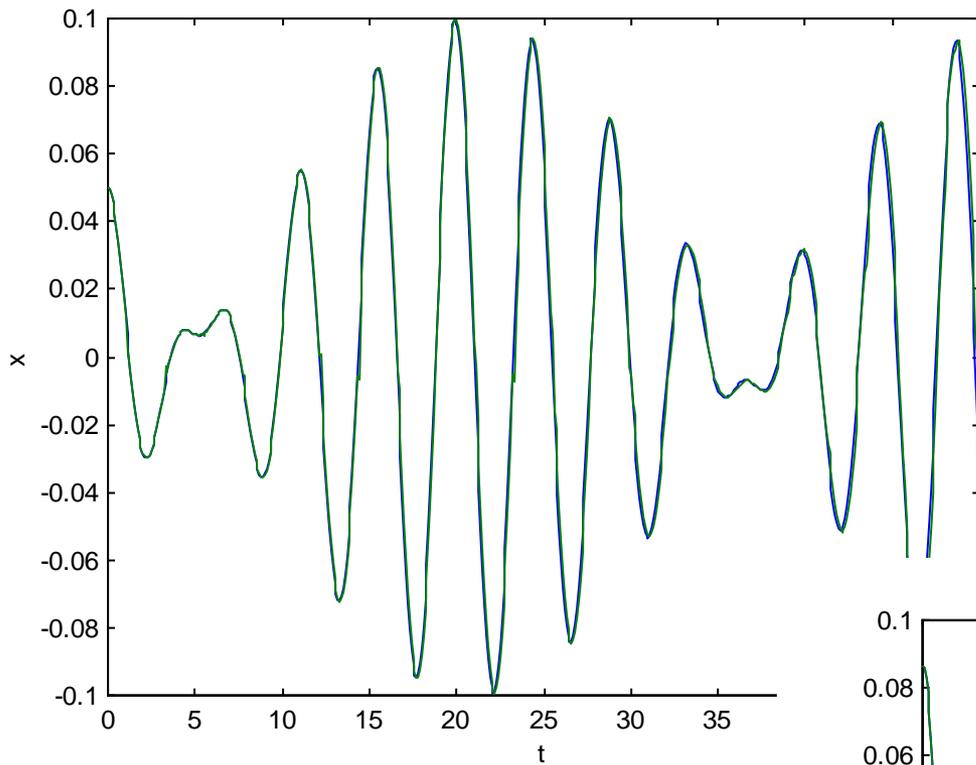
Runge Kutta numerički metod

$$\ddot{x} + 2x + x(x^2 + y^2) = -2\varepsilon g \dot{y},$$

$$\ddot{y} + 2y + y(x^2 + y^2) = 2\varepsilon g \dot{x}.$$

$$x(0) = 0.05, \quad \dot{x}(0) = 0,$$

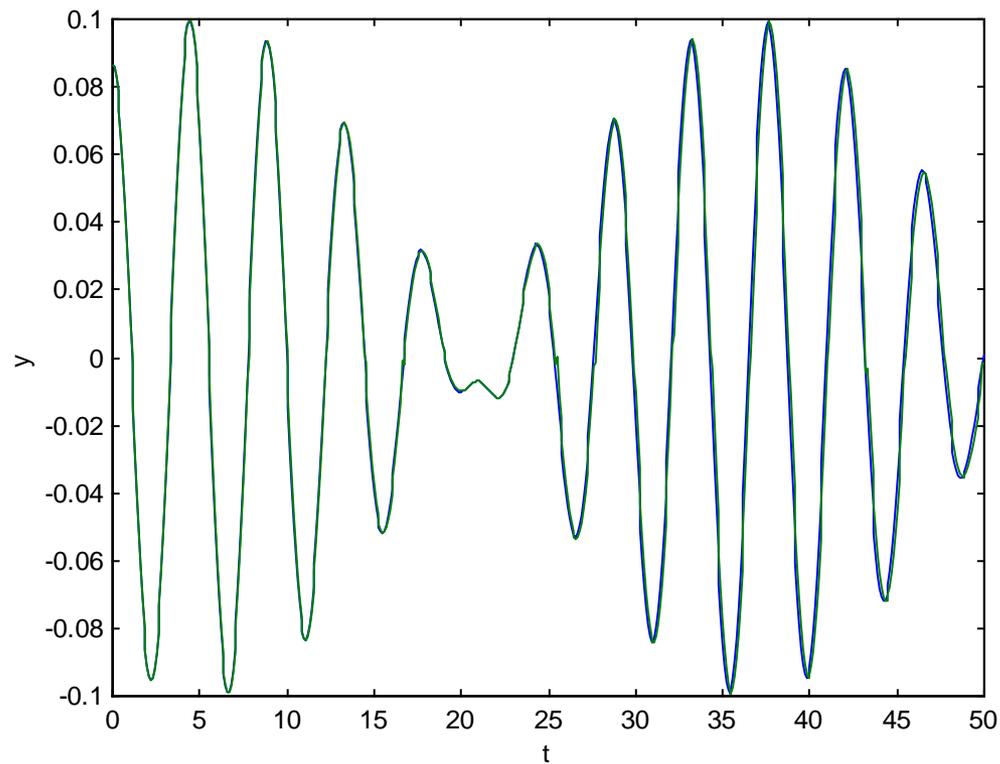
$$y(0) = 0.0866, \quad \dot{y}(0) = 0.$$



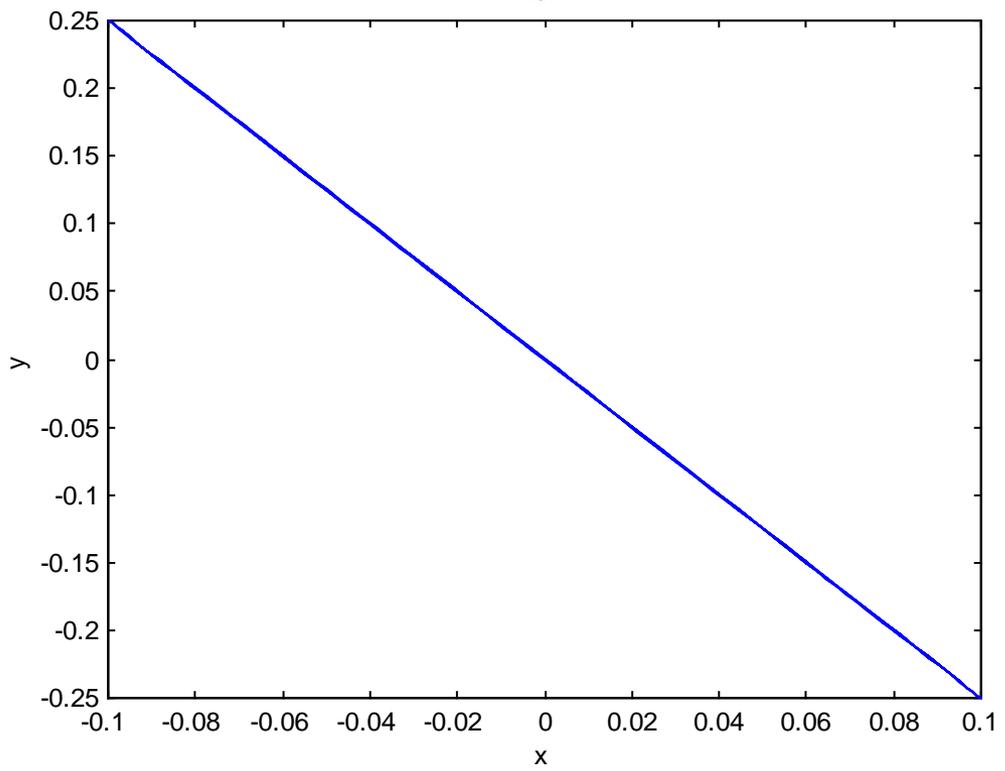
$$\rho(0) = 0.1,$$
$$\theta(0) = \pi/3,$$

$$\dot{\rho}_0 = 0,$$

$$\dot{\theta}_0 = 0.$$

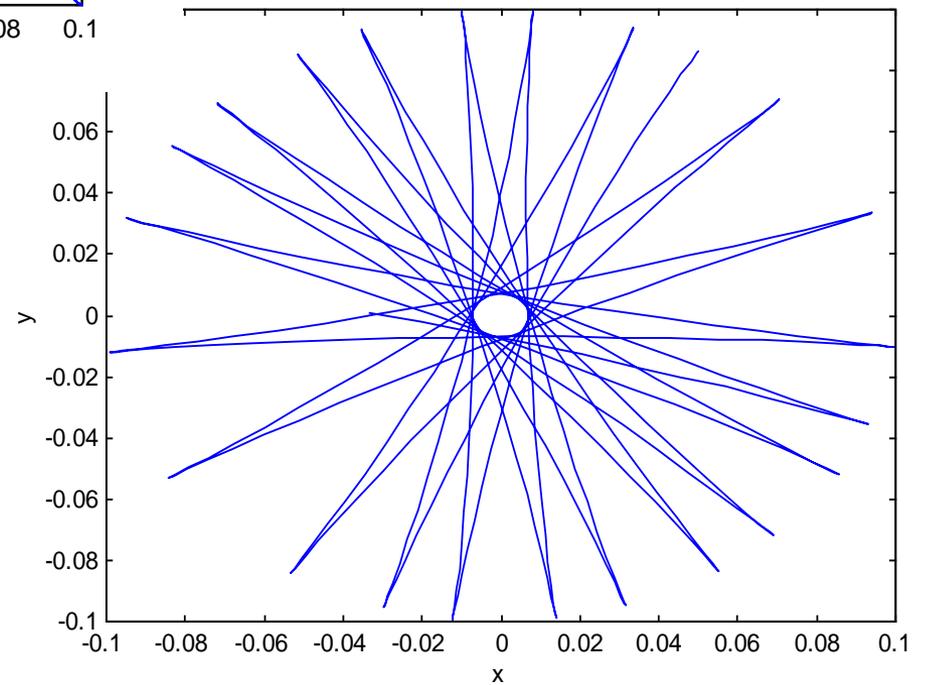


nisl1



$\varepsilon = 0$

$\varepsilon \neq 0$



# ZAKLJUČAK

- Metod prvih integrala ima prednost u odnosu na metod Krilov Bogoljubov jer daje tačnija rešenja.
- Drugi metod daje rešenje i za one slučajeve za koje prvi metod ne daje dovoljno tačne rezultate zbog osrednjavanja.
- Nedostatak metode prvih integrala je poteškoća oko integraljenja.

1.L. Cveticanin, Some particular solutions which describe the motion of the rotor, *Journal of Sound and Vibration*, Vol.212, No.1, pp.173-178, 1998.

2.L. Cveticanin, Analytic approach for the solution of the complex-valued strong non-linear differential equation of Duffing type, *Physica A*, Vol.297, No.3-4, pp. 348-360, 2001.

3.L. Cveticanin, Free vibration of a pure-cubic non-linear Jeffcott rotor, *Mechanism and Machine Theory*, accepted for publication in 2003.

-

## **5. Elastični metamaterijali i mogućnost njihove primene u savremenoj mašingradnji**

# BUKA

Negativni efekti buke:

- 30 dB psihički problemi
- 65 dB probleme sa vegetativnim nervnim sistemom
- 90 dB problemi sa sluhom
- 120 dB fizički bol
- 175 dB: uzrokuje smrt.

# Redukcija buke

- Frekvencije buke koje treba da budu absorbovane su u čujnom opsegu čoveka: od 100 Hz do 1 kHz
- Koriste se materijali za akustičku izolaciju.

# Transmisioni gubitak

Efikasnost izolacije - transmisioni gubitak  $TL$

$$TL = 10 \log_{10} \left( \frac{1}{T_c} \right)$$

gde je transmisioni koeficijent  $T_c$ .

$$TL = 10 \log_{10} \left[ 1 + \left( \frac{\pi \rho h f}{c \rho_a} \right)^2 \right]$$

$\rho$  je specifična gustina materijala,  $h$  debljina sloja,  $f$  je frekvencija,  $\rho_a$  je specifična gustina vazduha i  $c$  je brzina zvuka kroz vazduh.

# Rezultat

- Na datoj frekvenciji nivo redukcije buke za 5-6 dB ostvaruje se slojem udvajanjem sloja izolacije.
- Da bi povećao nivo gubitka pri transmisiji za 30 dB izolacioni sloj mora da se poveća toliko da mu se težina uveća 32-64 puta!

# Elektromagnetni metamaterijal

- Farba sa čeličnim kuglicama:
  - čelične kuglice električno izolovane karbonilom
  - specijalnih dimenzija raspodeljene po dvo-komponentalnoj epoksi farbi.
  - svaka od ovih mikroskopskih sfera obmotana je kvarcom (silikon dioksidom).

Specijalni kompozit

F-117A



# Elektromagnetni/Optički metamaterijal

To je veštački materijal, tipa kompozita, koji pri interakciji sa elektromagnetnim talasom dovodi do absorpcije talasa na određenoj frekvenciji.

Da li je po uzoru na  
elektromagnetni  
metamaterijal  
moguće napraviti novi  
materijal za redukciju buke  
i vibracija?

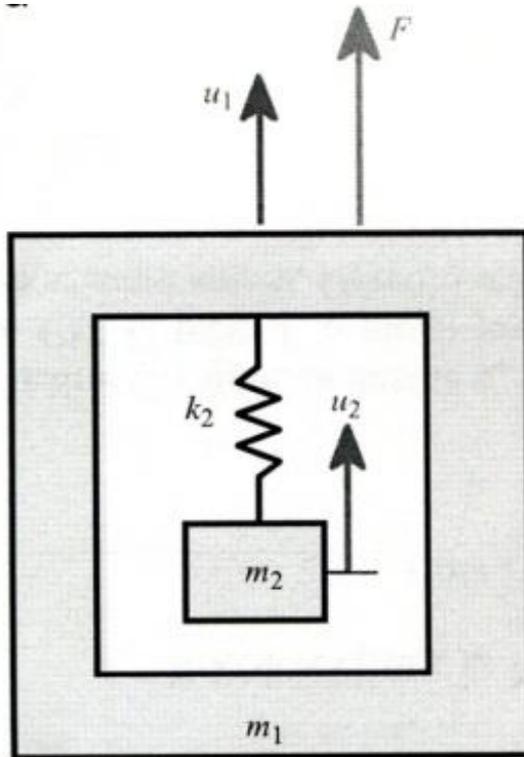
# Motivacija

- Zahvaljujući analogiji između elektromagnetnih i akustičkih talasa:
- 1. Elektromagnetni/optički metamaterijal koji zaustavlja elektro-magnetne talase
- 2. Akustički/elastični metamaterijal koji zaustavlja akustičke talase i eliminiše vibracije određenih frekvencija

# Osobine metamaterijala

- elektromagnetni metamaterijali imaju:
  - negativnu permitivnost  $\epsilon$
  - negativnu magnetnu permeativnost  $\mu$što rezultira negativnim indeksom refrakcije.
  
- Elastični metamaterijal treba da ima:
  - 'negativnu masu'  $m$
  - 'negativni modul elastičnosti'  $E$

# Sistem masa-u-masi



$$m_1 \ddot{u}_1 + k_2(u_1 - u_2) = F_0 \exp(i\omega t),$$

$$m_2 \ddot{u}_2 + k_2(u_2 - u_1) = 0.$$

# Rešenje jednačina

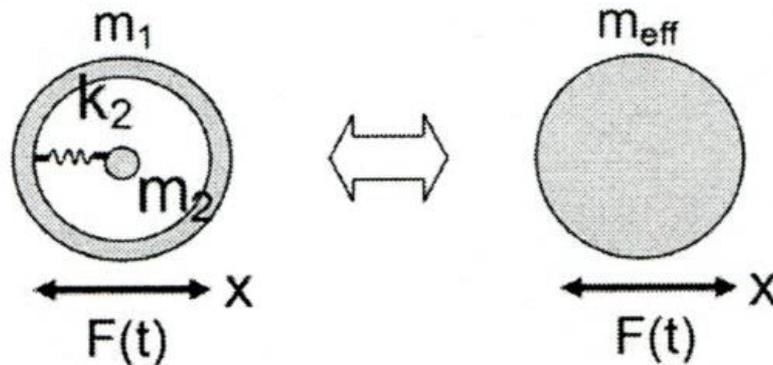
$$u_1 = a_1 \exp(i\omega t),$$

$$u_2 = a_2 \exp(i\omega t)$$

$$a_1 = \frac{F_0(k_2 - m_2\omega^2)}{(k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$a_2 = \frac{F_0k_2}{(k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

# Efektivna masa za Sistem masa-u-masi



Ekvivalentni model sistemu 'masa-u-masi' je jednomaseno telo sa efektivnom masom  $m_{eff}$  čije kretanje odgovara  $m_1$ .

Količina kretanja za oba  
modela su jednaka

$$m_{eff} \frac{du_1}{dt} = m_1 \frac{du_1}{dt} + m_2 \frac{du_2}{dt}$$

# Efektivna masa

$$m_{eff} = m_1 + m_2 \frac{k_2}{k_2 - m_2 \omega^2}$$

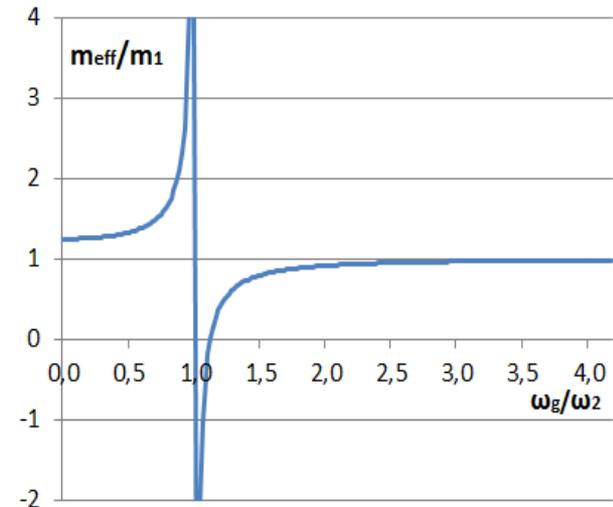
$$\text{Za } \omega_2 = \sqrt{k_2/m_2}$$

$$m_{eff} = m_1 + m_2 \frac{\omega_2^2}{\omega_2^2 - \omega^2}$$

# Efektivna masa

$$\text{Za } \omega_2 = \sqrt{k_2/m_2}$$

$$m_{eff} = m_1 + m_2 \frac{\omega_2^2}{\omega_2^2 - \omega^2}$$



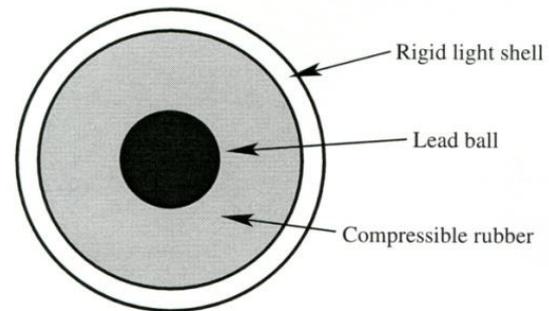
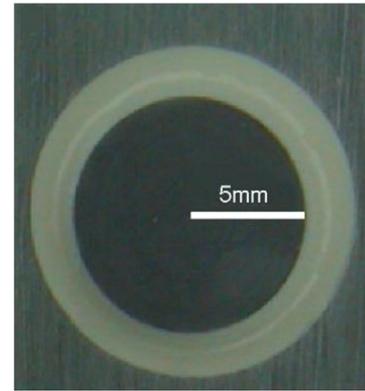
# Diskusija

1.  $\omega < \omega_2$  : akustički mod
  - efektivna masa  $m_{eff}$  je pozitivna
  - Kretanje obeju masa  $u_1$  i  $u_2$  je u fazi.
2.  $\omega > \omega_2$  : optički mod
  - efektivna masa  $m_{eff}$  može biti i pozitivna i negativna
  - Kretanje masa  $u_1$  i  $u_2$  je u suprotnoj fazi ( $180^\circ$ )

- Kako napraviti takvu strukturu koja će se ponašati kao da ima 'negativnu masu' a da osciluje na stalnoj frekvenciji iznad rezonantne?

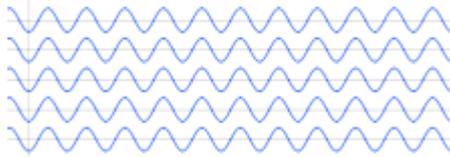
# Osnovni element metamaterijala

- sferno jezgro
- laka, ljuska za obmotavanje
- guma ili podesni anizotropni materijal (silikon, bizmut)

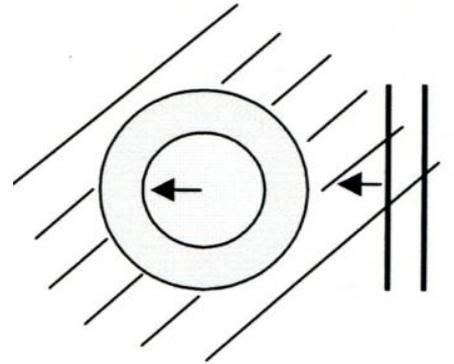


# Interakcija jezgra i sredine

- Jezgro osciluje u fazi sa omotačem i talasom ( $\omega < \omega_2$ ),
- efektivna masa je pozitivna



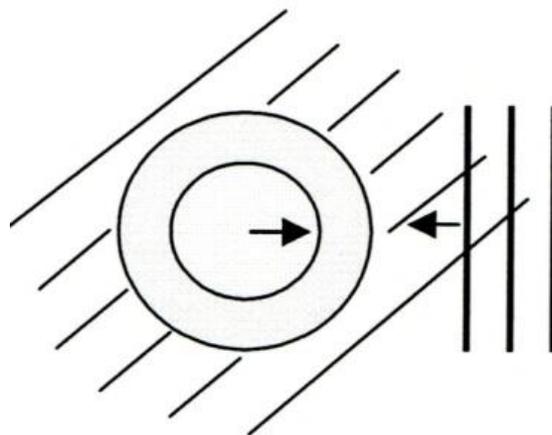
Talasi u fazi



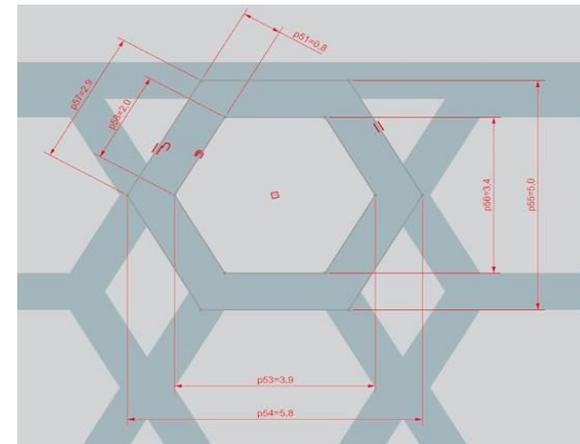
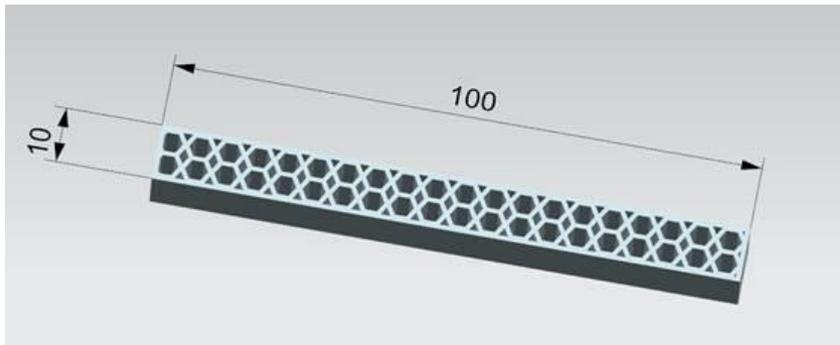
- Za  $\omega > \omega_2$  jezgro osciluje sa fazom koja je suprotna od faze talasa i omotača.
- Efektivna masa je negativna: gustina nanosa je dovoljno debela.

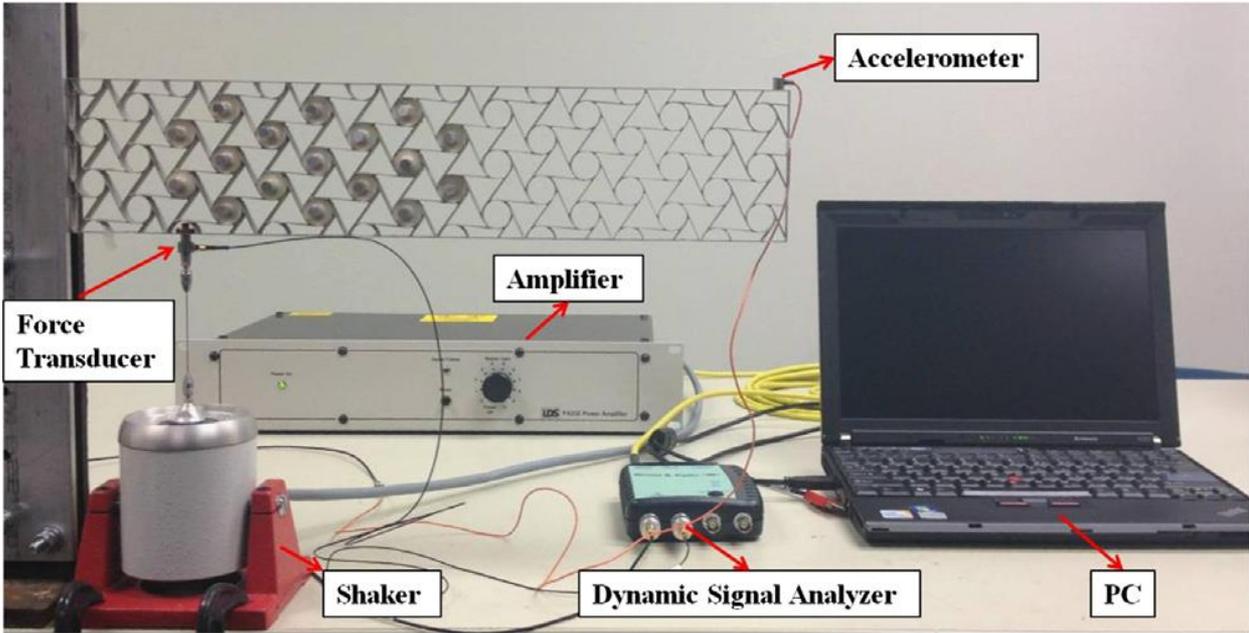


Talasi sa suprotnom fazom



# Dvoredi štap od elastičnog metamaterijala

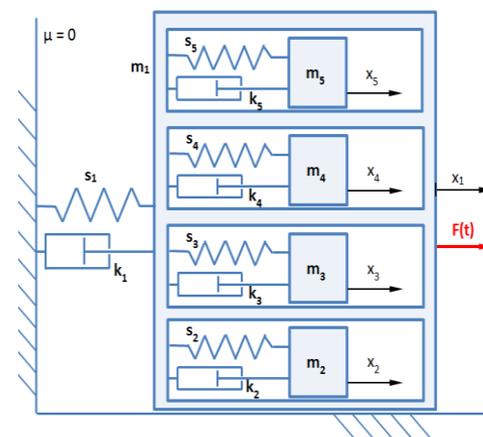
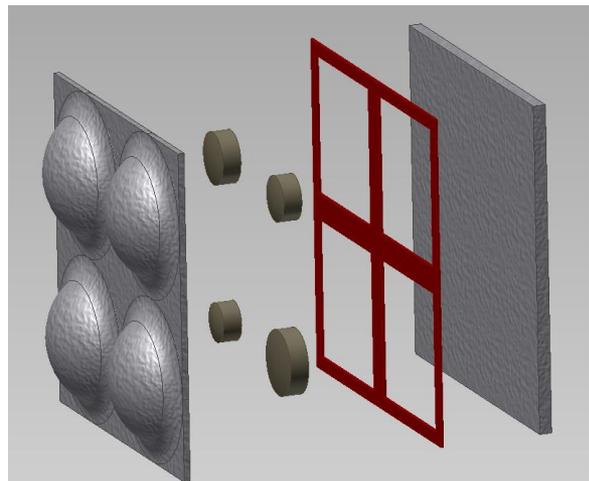




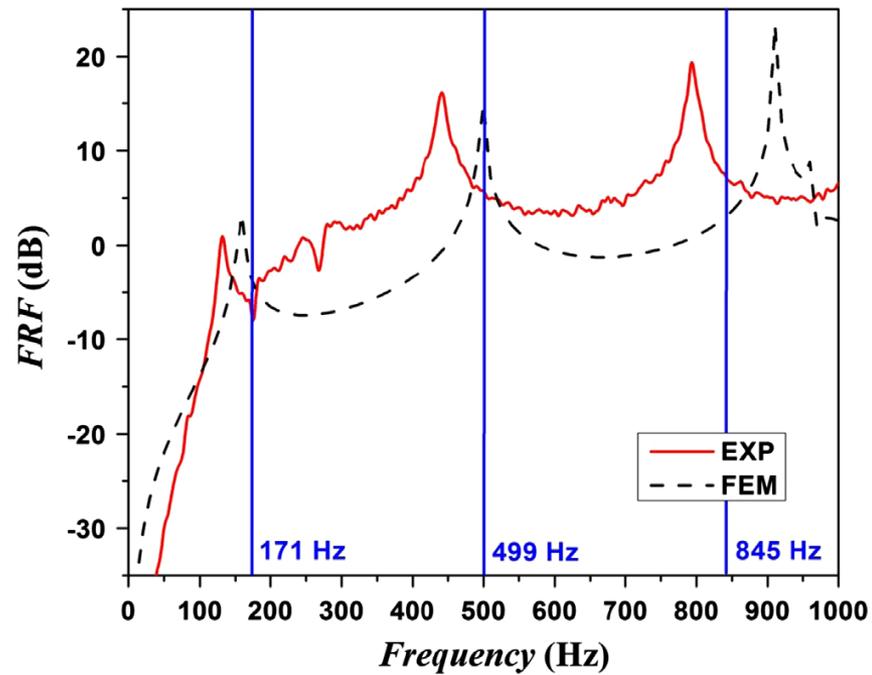
# Model od metamaterijala u vidu ploče

Između **pl** **oče** **pl** **oče** ploče postavljeni su absorberi.

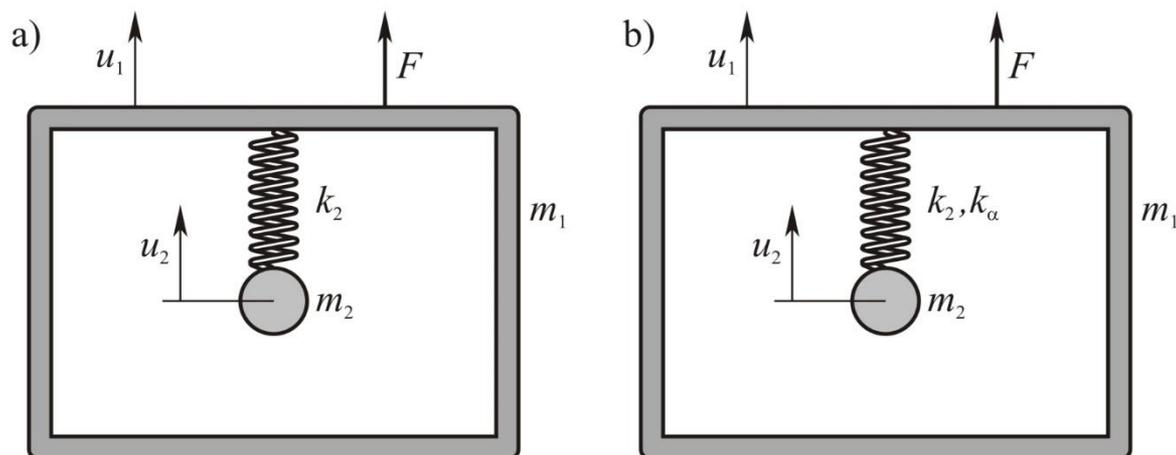
Odgovarajućim razmeštajem elemenata spoljašnja ploča nema vibracije na određenoj frekvenciji.



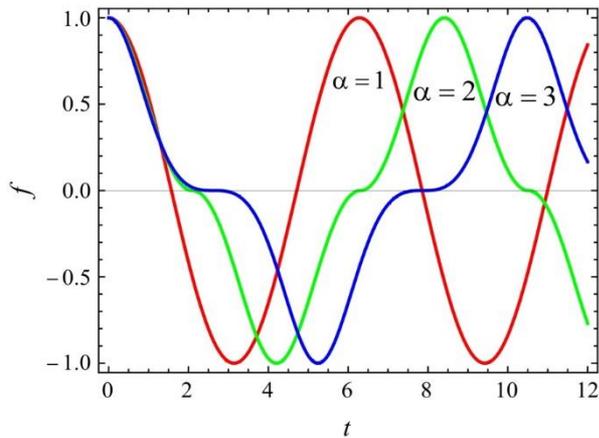
FRF (odnos izlaznog naponskog signala u akcelerometru i ulaznog napona pobude) u funkciji od frekvencije



# Koncept nelinearnog absorbera



# Uopštavanje modela: Pobudna sila multiharmonijska



$$F = \sum_{n=1}^{\infty} f \cos(n\pi\Omega t)$$

$$\sum_{n=1}^{\infty} a_n \cos \frac{n\pi\Omega t}{\Pi_\alpha} = ca(\alpha, 1, \Omega t)$$

# Ateb funkcija

- Period:

$$T = 2\Pi_{\alpha} = 2B\left(\frac{1}{\alpha+1}, \frac{1}{2}\right),$$

- gde je B - Beta funkcija koja zavisi od  $\alpha$ .

# Nelinearni model

$$m_1 \ddot{u}_1 + k_\alpha^2 (u_1 - u_2) |u_1 - u_2|^{\alpha-1} = Fca(\alpha, 1, \Omega t) |ca(\alpha, 1, \Omega t)|^{\alpha-1},$$

$$m_2 \ddot{u}_2 + k_2^2 (u_2 - u_1) |u_1 - u_2|^{\alpha-1} = 0.$$

Rešenje:

$$u_1 = Aca(\alpha, 1, \Omega t), \quad u_2 = Bca(\alpha, 1, \Omega t),$$

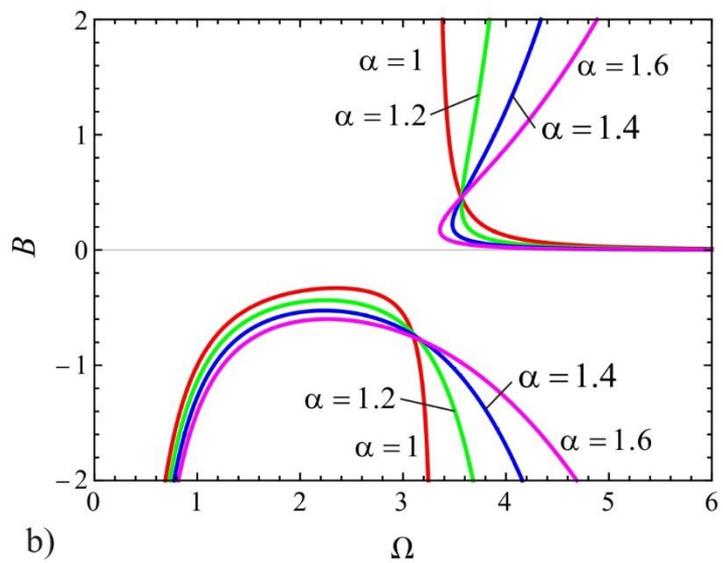
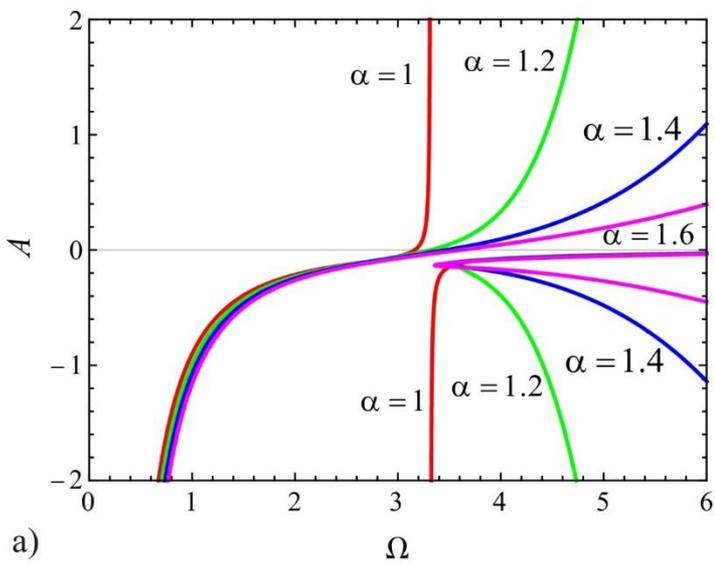
$$m_1 A + m_2 B = -F \frac{\alpha + 1}{2\Omega^2},$$

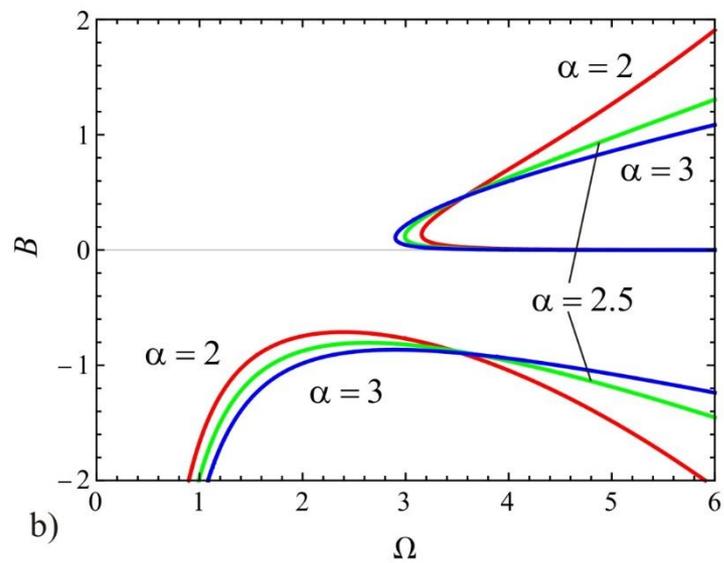
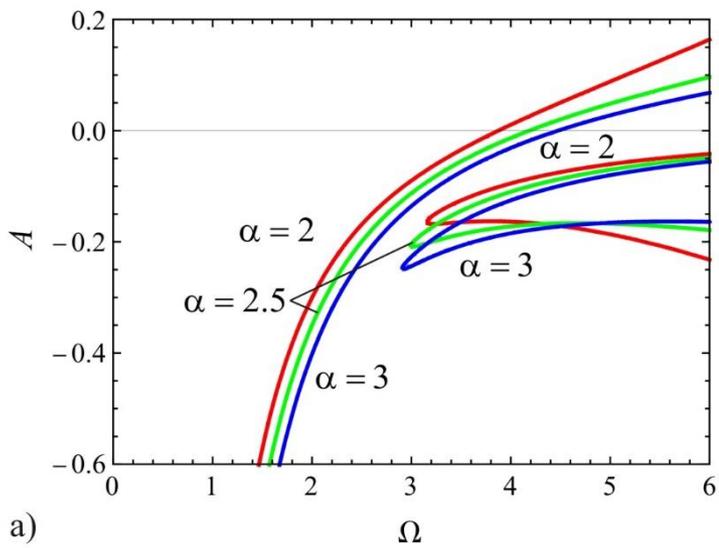
$$-m_2 \Omega^2 \frac{2}{\alpha + 1} B + k_\alpha^2 (B - A) |A - B|^{\alpha - 1} = 0.$$

• odnosno

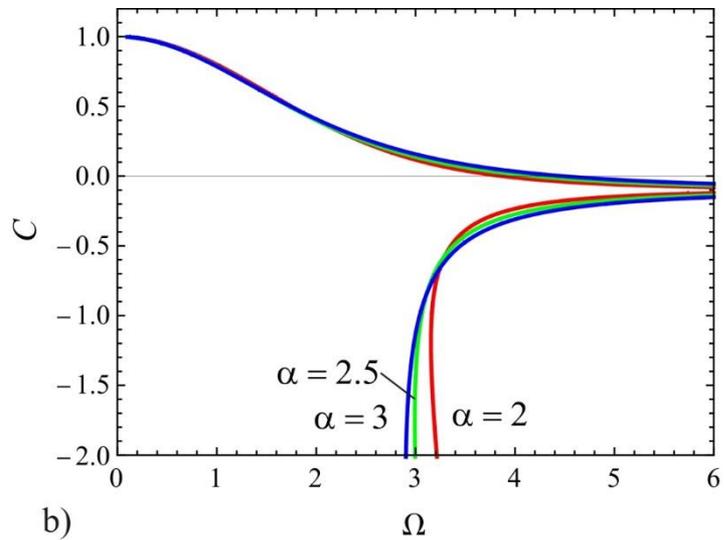
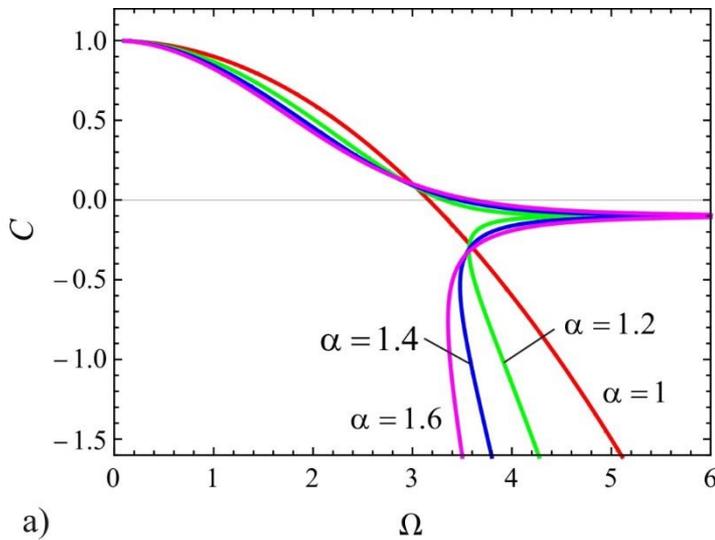
$$m_1 \frac{2\Omega^2}{\alpha + 1} A - k_\alpha^2 \left( \frac{m_1 + m_2}{m_2} A + F \frac{\alpha + 1}{2\Omega^2 m_2} \right) \left| \frac{m_1 + m_2}{m_2} A + F \frac{\alpha + 1}{2\Omega^2 m_2} \right|^{\alpha - 1} = -F.$$

$$m_2 \frac{2\Omega^2}{\alpha + 1} B - k_\alpha^2 \left( \frac{m_1 + m_2}{m_1} B + F \frac{\alpha + 1}{2\Omega^2 m_1} \right) \left| \frac{m_1 + m_2}{m_1} B + F \frac{\alpha + 1}{2\Omega^2 m_1} \right|^{\alpha - 1} = 0.$$



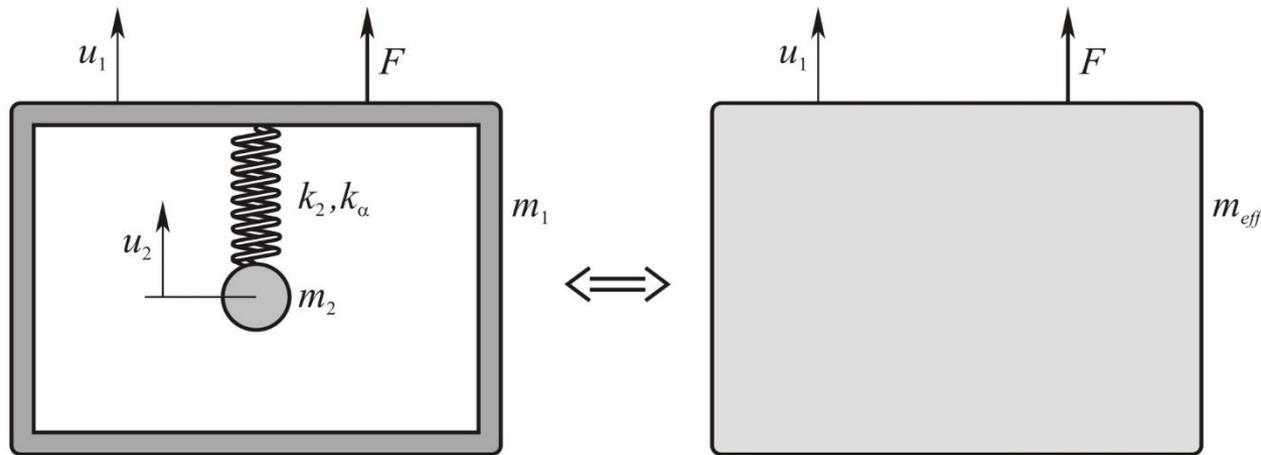


$$(2\Omega^2)^\alpha m_2 + k_\alpha^2 (C-1)(\alpha+1)^\alpha \left| \frac{F(C-1)}{m_1 C + m_2} \right|^{\alpha-1} = 0.$$



$$C = \frac{A}{B}, \quad C_\infty = -\frac{m_2}{m_1}, \quad \frac{k_\alpha^2}{m_2^\alpha} = \frac{1}{F^{\alpha-1}} \left( \frac{2\Omega^2}{\alpha+1} \right)^\alpha$$

# Koncept efektivne mase



$$m_{eff} \dot{u}_1 = m_1 \dot{u}_1 + m_2 \dot{u}_2.$$

# Efektivna masa

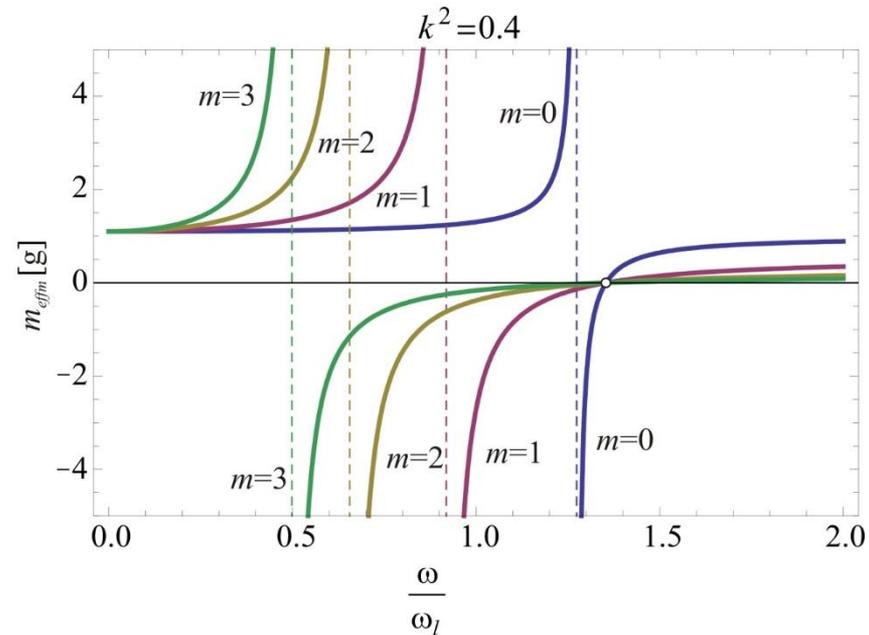
$$m_{eff} = m_1 + \frac{m_2}{C}.$$

$$m_{eff} = m_1 + m_2 + \frac{m_2 \left( \frac{2\Omega^2}{\alpha + 1} \frac{m_2}{k_\alpha^2 B^{\alpha-1}} \right)^{1/\alpha}}{1 - \left( \frac{2\Omega^2}{\alpha + 1} \frac{m_2}{k_\alpha^2 B^{\alpha-1}} \right)^{1/\alpha}},$$

# Negativna efektivna masa

$$1 < \frac{\Omega}{\Omega_1} < \left(1 + \frac{m_2}{m_1}\right)^{\alpha/2}.$$

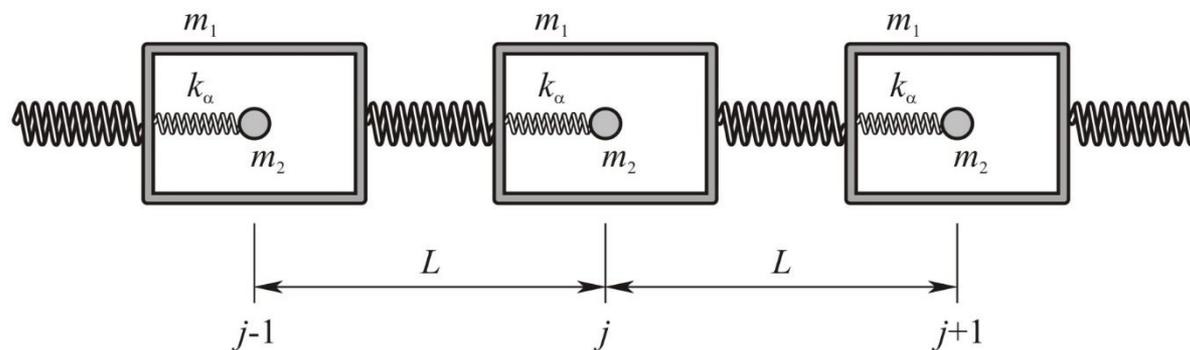
$$\Omega_1^2 = \frac{k_\alpha^2 B^{\alpha-1}}{2m_2} (1 + \alpha),$$



L. Cveticanin, M. Zukovic, Negative effective mass in acoustic metamaterial with nonlinear mass-in-mass subsystems, *Communications in Nonlinear Science and Numerical Simulation*, 51, 89-104, 2017.

# Akustički metamaterijal - Jednodimenzionalni lanac

- 

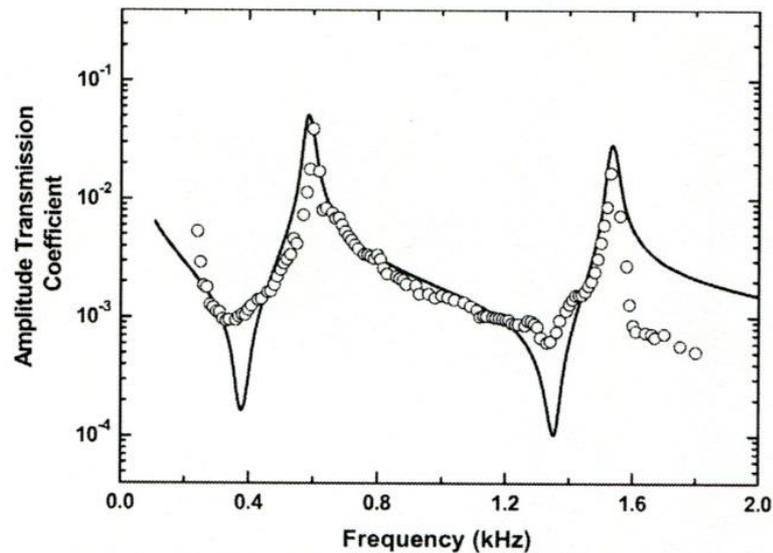


# Efektivna masa

- Veza je linearna

- $$m_{eff} = m_1^{(j)} + m_2^{(j)} + \frac{m_2^{(j)} \left( \frac{2\Omega^2 m_2^{(j)}}{\alpha+1 k_\alpha^2 B^{\alpha-1}} \right)^{1/\alpha}}{1 - \left( \frac{2\Omega^2 m_2^{(j)}}{\alpha+1 k_\alpha^2 B^{\alpha-1}} \right)^{1/\alpha}}$$

Merene (krugovi) i  
sračunate (puna linija)  
vrednosti u funkciji od  
frekvencije



# Zaključak

- Kada nelinearni sistem ima negativnu efektivnu masu, frekvencija koja odgovara negativnoj masi predstavlja zaustavnu regiju.
- Ova oblast je najuža za linearni sistem.
- Širina te trake se povećava povećanjem stepena nelinearnosti elastičnog elementa.
- Otuda, nelinearni oscilatorni sistem ima prednosti u odnosu na linearni, jer je zaustavna oblast frekvencija šira.

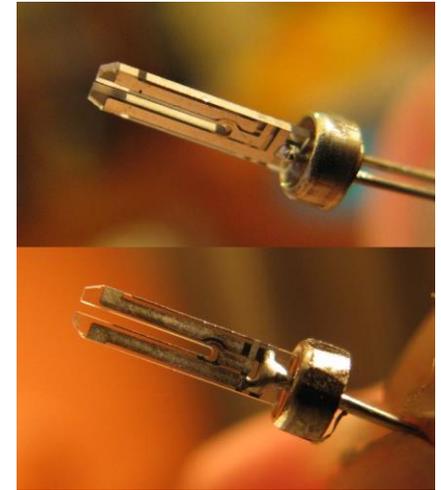
# NOVA ISTRAŽIVANJA

- Elastični metamaterijal je dokazano da je efikasan za eliminisanje vibracija na određenoj frekvenciji.
- Da li je moguće da se vibraciona energija metamaterijala transformiše u električnu energiju?

# UREĐAJI POGONJENI MALOM ENERGIJOM

- Mikro-elektro-mehanički sistemi MEMS (senzori, aktuatori)
- Merni i kontrolni sistemi
- Male električne komponente i senzori kod:
  - Mašina
  - Letelica
  - Električnih postrojenja
  - Građevinarstvu

- Električni uređaji malih dimenzija (mobilni i fiksni)
  - Bežično napajanje
  - Kontinualno napajanje (energetski harvesteri)
- Napajanje spolja nije moguće



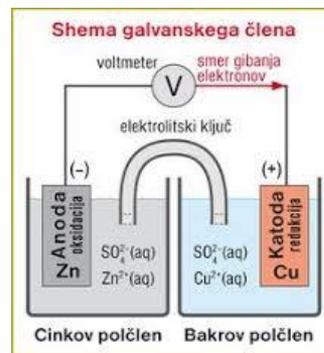
# Napajanje baterijama

- **Prednosti**

- Efikasne
- Niska cena

- **Nedostaci**

- Ograničen vek trajanja
- Zagađivači okoline
  - Hemijski otpad

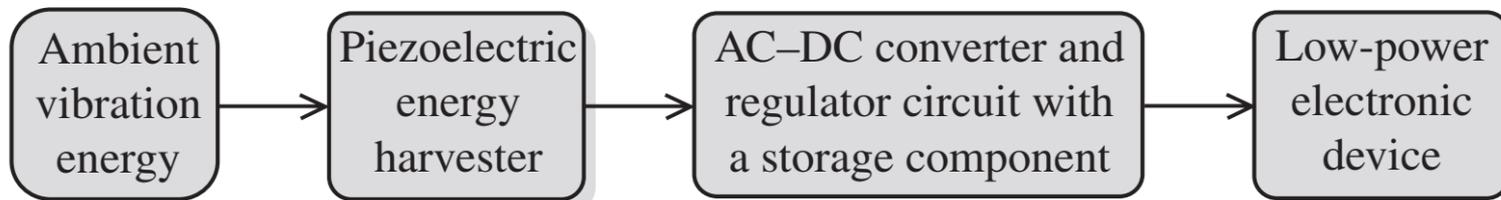


# ZAHTEVI PRI PROJEKTOVANJU MALIH IZVORA ENERGIJE

- Obnovljivost
- Primena ambijentalne energije
  - Vazдушna struja
  - Promena pritiska
  - Ljudska snaga
  - Vibracije

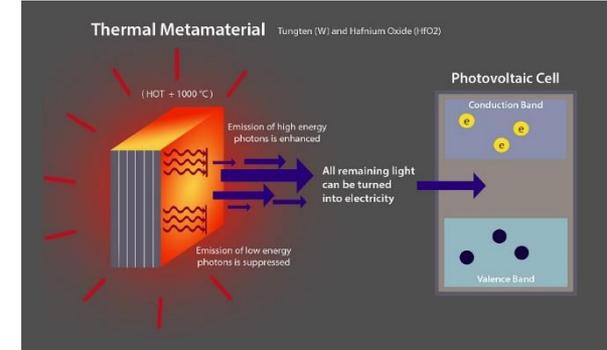
# ENERGETSKI HARVESTERI (EH)

- Uređaji koji transformišu jedan vid energije u drugi a vrši i skladištenje energije.



# ENERGETSKI HARVESTERI

## PRIMERI



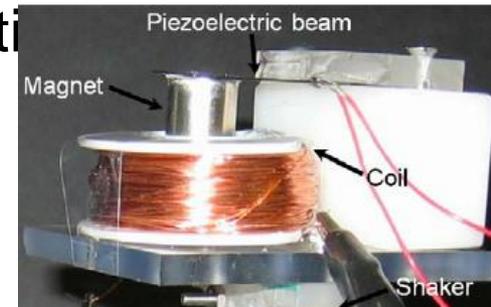
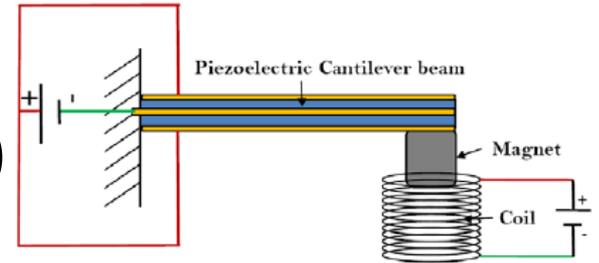
- Termalni EH konvertuje toplotnu energiju u mehanički rad.
  - Temperature promene uzrokuju kretanje molekula i energija se transformiše.
- Foto-naponski EH konvertuje svetlost u elektricitet na atomskom nivou.
  - Foto-električni efekat materijala: absorbuje fotone i pomera elektrone koji stvaraju struju.

# MEHANIČKA U ELEKTRIČNU ENERGIJU

- Kinetički EH
  - Kinetička energija kretanja se transformiše u električnu.
- Vibracioni EH
  - Mehanička energija vibracija se konvertuje u električnu energiju.

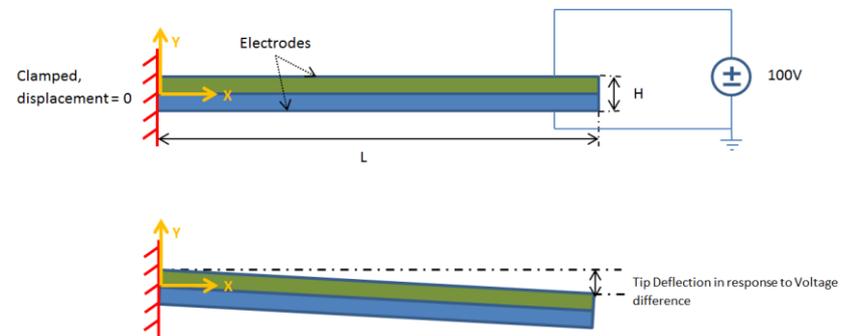
# 3 TIPRA GENERATORA

- Elektromagnetni generator koristi:
  - Elektromagnetna indukcija koja je rezultat relativnog kretanja između provodnika i magneta sa promenljivim fluksom.
  - Pogodan za male masa-opruga konfiguracije.
  - **Prednost:** Nisko-naponski elektricitet i velika izlazna struja
  - **Nedostatak:** Problemi u spajanju i u saosnosti u elektromagnetskom sistemu sa sub-milimeterskom dimenzijom.



# ELEKTROSTATIČKI GENERATOR

- Elektrostatički generator koristi relativno kretanje između električno izolovanih naelektrisanih ploča kondenzatora.
- Rad elektrostatičke sile između ploča daje energiju koja se skladišti.



- **PREDNOST:**

- Gustina energije raste smanjenjem razmaka između ploča i povećanjem površine ploča kondenzatora.
- Elektrode daju inicijalno naelektrisanje i višegodišnje skladišno naelektrisanje.

- **NEDOSTACI:**

- Izlazni napon je relativno visok.
- Snabdevanje i kapacitivnost je limitirana redukcijom efikasnosti generatora.

# PIEZOELECTRIČNI GENERATORI

Piezoelektrični generator konvertuje direktno strukturne vibracije u električnu energiju korišćenjem piezoelektričnog materijala.

Piezoelektrični materijal generiše naelektrisanje usled deformisanja. Materijal je veštački i od kompozita.

- **PREDNOST:**
- Jednostavna konstrukcija
  - Nema zahteva za složenom geometrijom i brojnim dodatnim elementima
- Mala veličina

Daje relativno visok izlazni napon i malu struju.

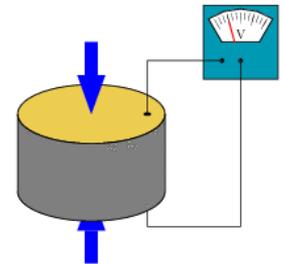


# PRONALAZAČI PIEZOEFEKTA

- Pierre i Jacques Curie, 1880.

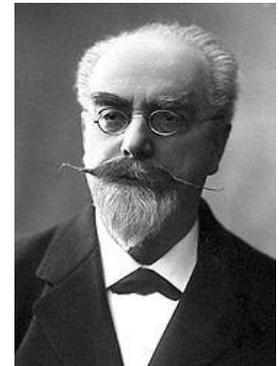
Ako se kristalni materijal (topaz, kvarc,...) pritisnu, generiše se električna energija.

Na grčkom: Piezein - pritisnuti



# 'INVERZNI PIEZOELEKTRIČNI EFEKAT'

- Efekat je matematički potvrdio Gabriel Lippman 1881. godine
- Primena: kod piezoelektričnih ultrasoničnih davača u I svetskom ratu.

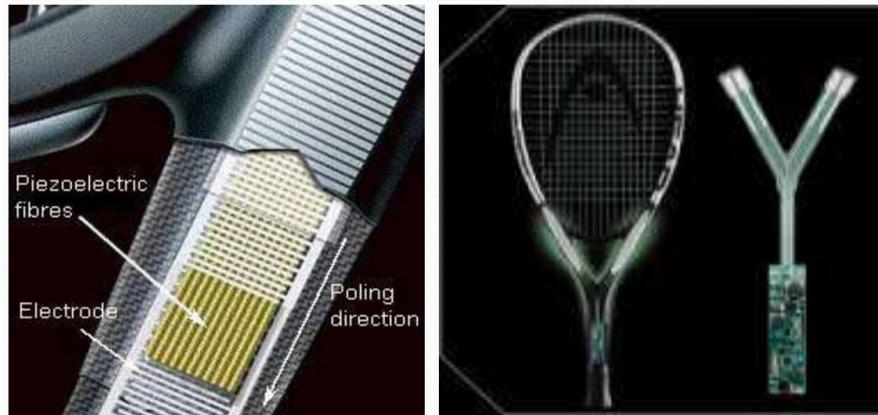


# MATERIJALI ZA PIEZOEFEKAT

- KRISTALI:
  - Kvarc (silicijum-dioksid)
    - Satovi
    - 1967, Neuchatel, Švajcarska
    - Mehanička energija kretanja ruke se transformiše u električnu energiju.
  - Topaz
  - Saharoza
  - ...



- Korak može da da električnu struju koja je dovoljna za 2 sijalice da svetle 1 sekundu.

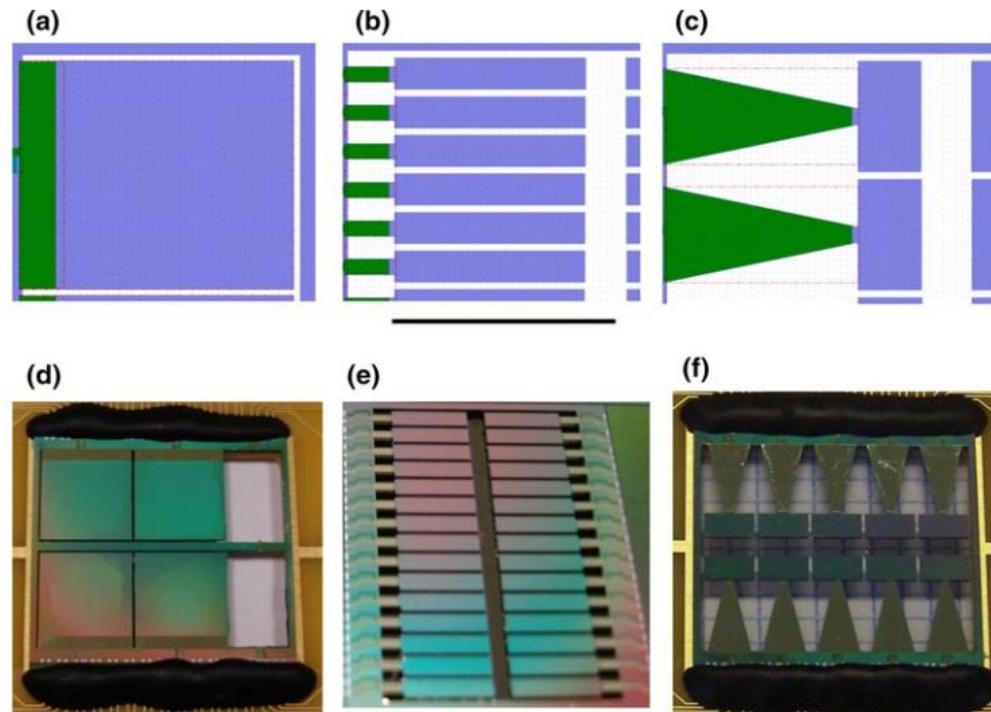


# TIPOVI PIEZOELEMENTATA

- ŠIROKI
- TANKI
- TRAPEZOIDNI ŠTAP

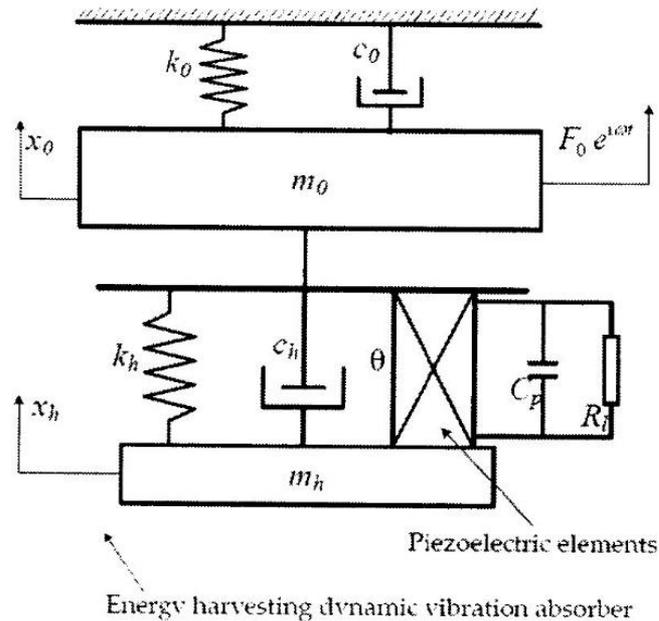
Zeleno – štap

Ljubičasto – masa



- Da bi se to postiglo vibracioni apsorber je snabdeven piezoelektričnim elementom za:
  - Ograničavanje vibracija i
  - Harvesting energije
  - - Originalna struktura je osetljiva na vibracije
  - - Absorber je element koji ograničava vibracionu energiju
  - - Piezoelektrični element harvestuje energiju

# Model EH sa metamaterijalom



# MATRMATIČKI MODEL

$$m_0\ddot{x}_0 + c_0\dot{x}_0 + k_0x_0 - k_h(x_h - x_0) - c_h(\dot{x}_h - \dot{x}_0) = F_0e^{i\omega t} \quad (1)$$

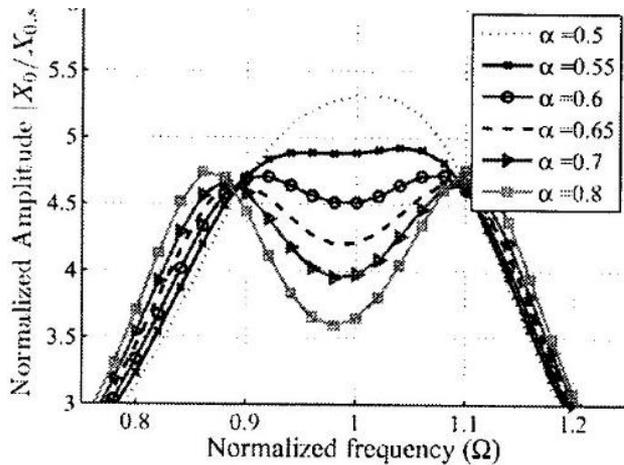
$$m_h\ddot{x}_h + c_h(\dot{x}_h - \dot{x}_0) + k_h(x_h - x_0) - \theta v = 0 \quad (2)$$

$$C_p\dot{v} + \frac{v}{R_l} + \theta\dot{x}_h = 0 \quad (3)$$

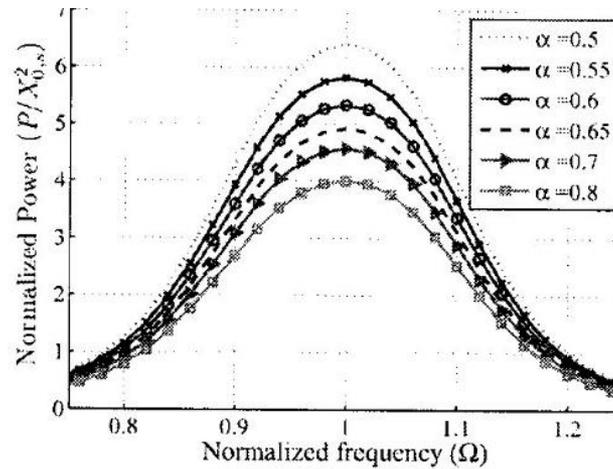
- V – NAPON
- $C_p$  – KAPACITIVNOST
- RI - OTPOR

# Snaga u EH

$$P = \frac{|V|^2}{R_l} = \frac{|V|^2}{\alpha} C_p \omega_h$$



(a)



(b)

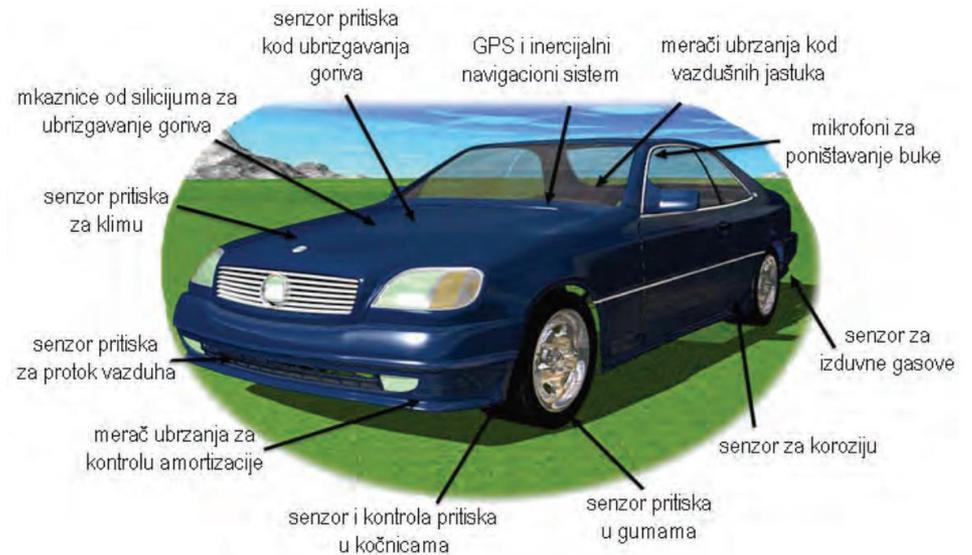
# DALJA ISTRAŽIVANJA

- Primena elastičnog metamaterijala za električni harvester.
  - Podesiti frekventni opseg
  - Optimizirati odnos između energije skladištenja i redukcije vibracija.

# PRIMENA U EH

- ELEKTROAKUSTIKA

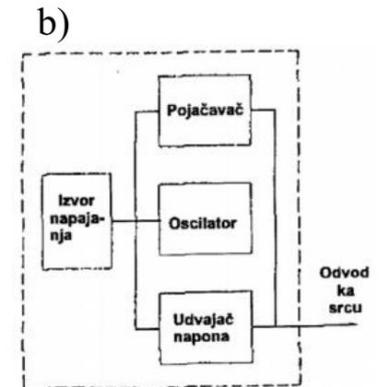
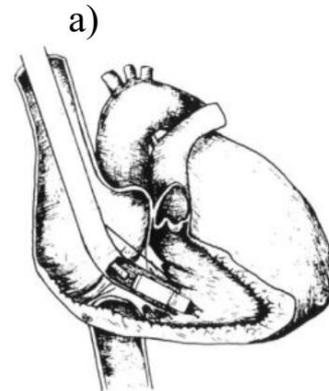
- ▶ MIKRO-ELEKTRO-MEHANIČKI SISTEMI (MEMS)



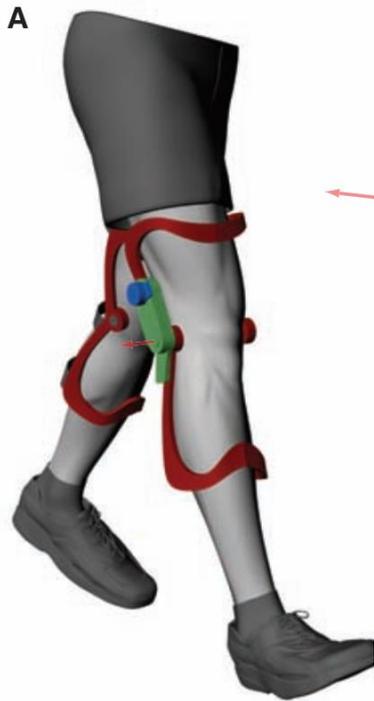
*Primena MEMS u automobilu.*

# PRIMENA U MEDICINI

- ▶ Implanti:
- ▶ Pejsmejkeri
  - ▶ Više od 20 godina
  - ▶ Zamenjuju srčani mišić - automatski daje regularne impulse i kontrakcije
  - ▶ Uređaj u gornjoj desnoj komori



- Biomehanika EH

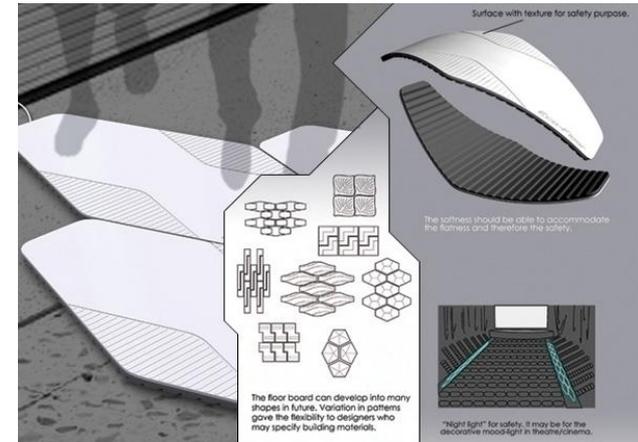


Medicinski uređaji za  
kontinualni monitoring



# PRIMENA U GRAĐEVINARSTVU

- ▶ Eko-Energetski Pokrivni Sistem
  - ▶ Pod sa 6 slojeva(plastilni, OLED table, piezoelektrični slojr, “papirni sloj” baterija)
  - ▶ Ploče se postave na pešačkom šetalištu
  - ▶ Električna struja za osvetljenje, reklame...

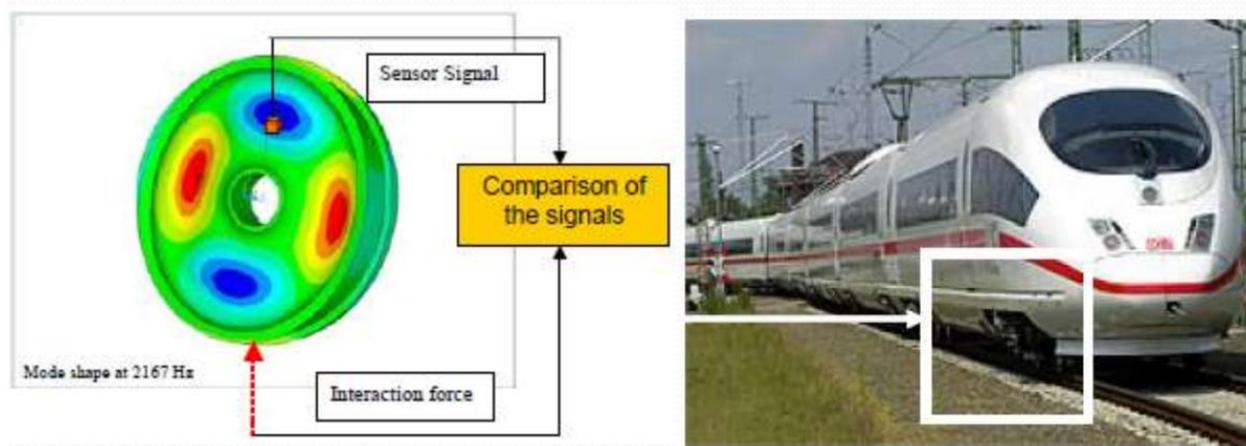


- “Pametni putevi” i mostovi



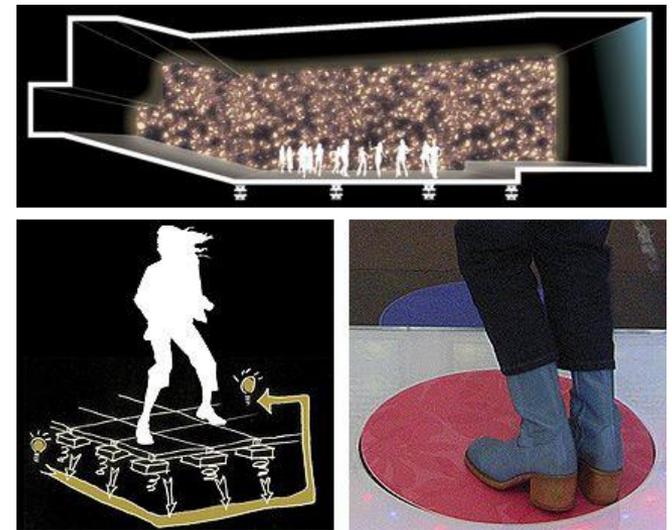
# PRIMENA U SAOBRAĆAJU

- Senzori za otkrivanje promena  
Vibracije na točkovima voza



# PRIMENA NA ZABAVI

- “Piezoelektrični pod”
- Klub za 3 noći potroši energiju koja je 150 puta veća nego jednogodišnja potrošnja četvoročlane porodice



# Reference

- L. Cveticanin, M. Zukovic, D. Cveticanin, Non-ideal source and energy harvesting, *Acta Mechanica*, 228(10), 3369-3379, 2017.
- L.Cveticanin, M. Zukovic, D. Cveticanin, On the elastic metamaterial with negative effective mass, *Journal of Sound and Vibration*, 436, 295-309 2018.
- L. Cveticanin, M. Zukovic, J.M. Balthazar, Non-ideal energy harvester with piezoelectric coupling, in *Dynamics of Mechanical Systems with Non-Ideal Excitation*, Mathematical Engineering, Springer, 9783319541686, 173-219, 2018.
- L. Cveticanin, M. Zukovic, D. Cveticanin, Influence of nonlinear subunits on the resonance frequency band gaps of acoustic metamaterial, *Nonlinear Dynamics*, 93(3), 2018.
- L. Cveticanin, M. Zukovic, D. Cveticanin, Steady state vibration of the periodically forced and damped pure nonlinear two-degrees-of-freedom oscillator, *Journal of Theoretical and Applied Mechanics*, (PTMTS), Vol.57, No.2, 445-460, 2019.

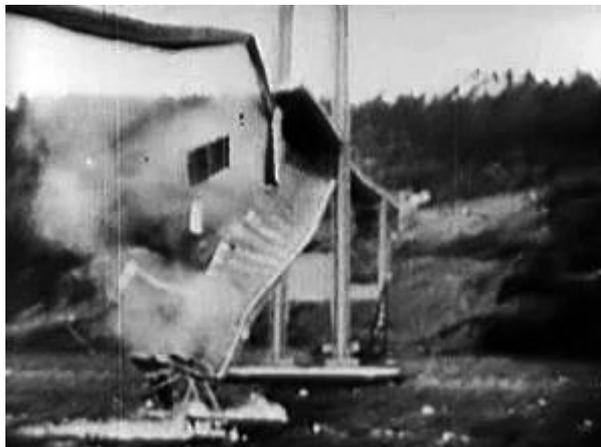
# 6. DETERMINISTICKI HAOS

# Tacoma Narrows Bridge

1. juli 1940. – 7. november 1940.



# Uzroci kolapsa

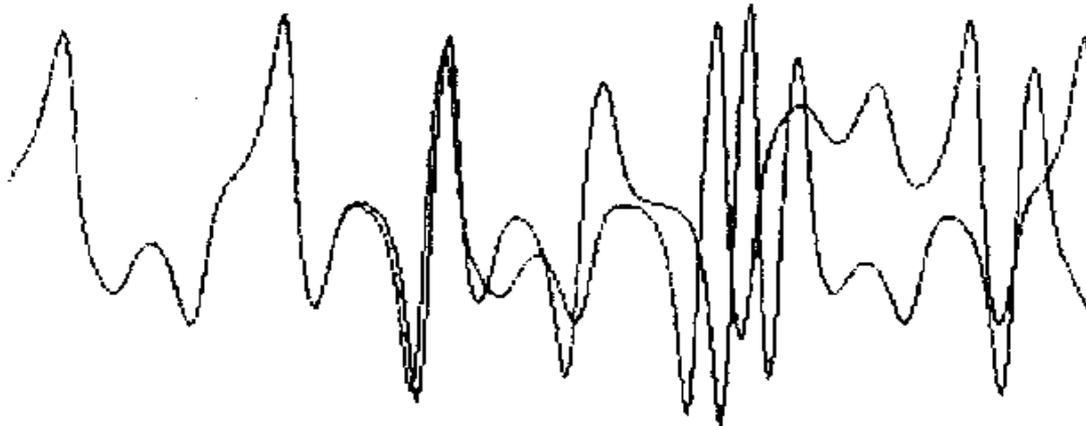


- Turbulencija i rezonanca
- Periodicni vrtlozi I torziona oscilacije
- Aerodinamicka nestabilnost i samopobudne vibracije

# MIT meteorolog Edward Lorenz



Otac teorije haosa  
i 'leptir efekta'



*Figure 1: Lorenz's experiment: the difference between the start of these curves is only .000127. (Ian Stewart, Does God Play Dice? The Mathematics of Chaos, pg. 141)*



# “Butterfly Effect”

Naziv efekta je usvojen po naucnom radu koji je objavio 1972:

"Predictability:

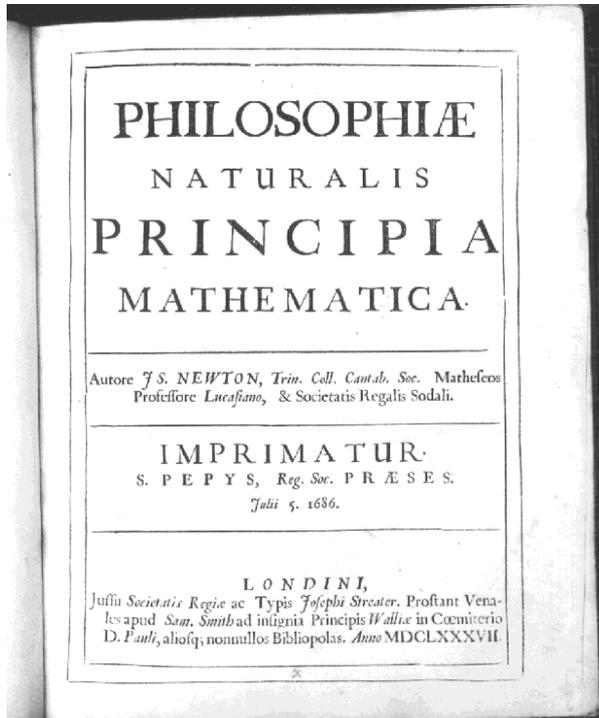
Does the Flap of a Butterfly's Wings in

Brazil Set Off a Tornado in Texas?"



# Deterministicki haos

- Χαος – prazan prostor



- Isaac Newton (1642-1727)



# Henri Poincaré (1854-1912)

" (...) si ce joueur bat les cartes assez longtemps, il y aura un grand nombre de permutations successives ; et l'ordre final qui en résultera ne sera plus régi que par le hasard ; je veux dire que tous les ordres possibles seront également probables. "

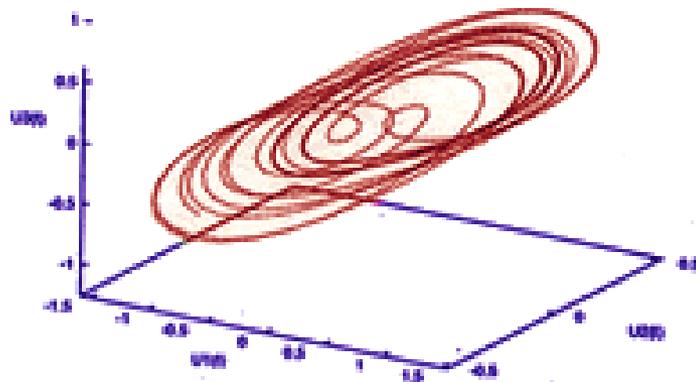


**POINCARÉ (H.)** ; Science et Méthode. Paris, Ernest Flammarion 1908.

# Da li postoji haos u mozgu?

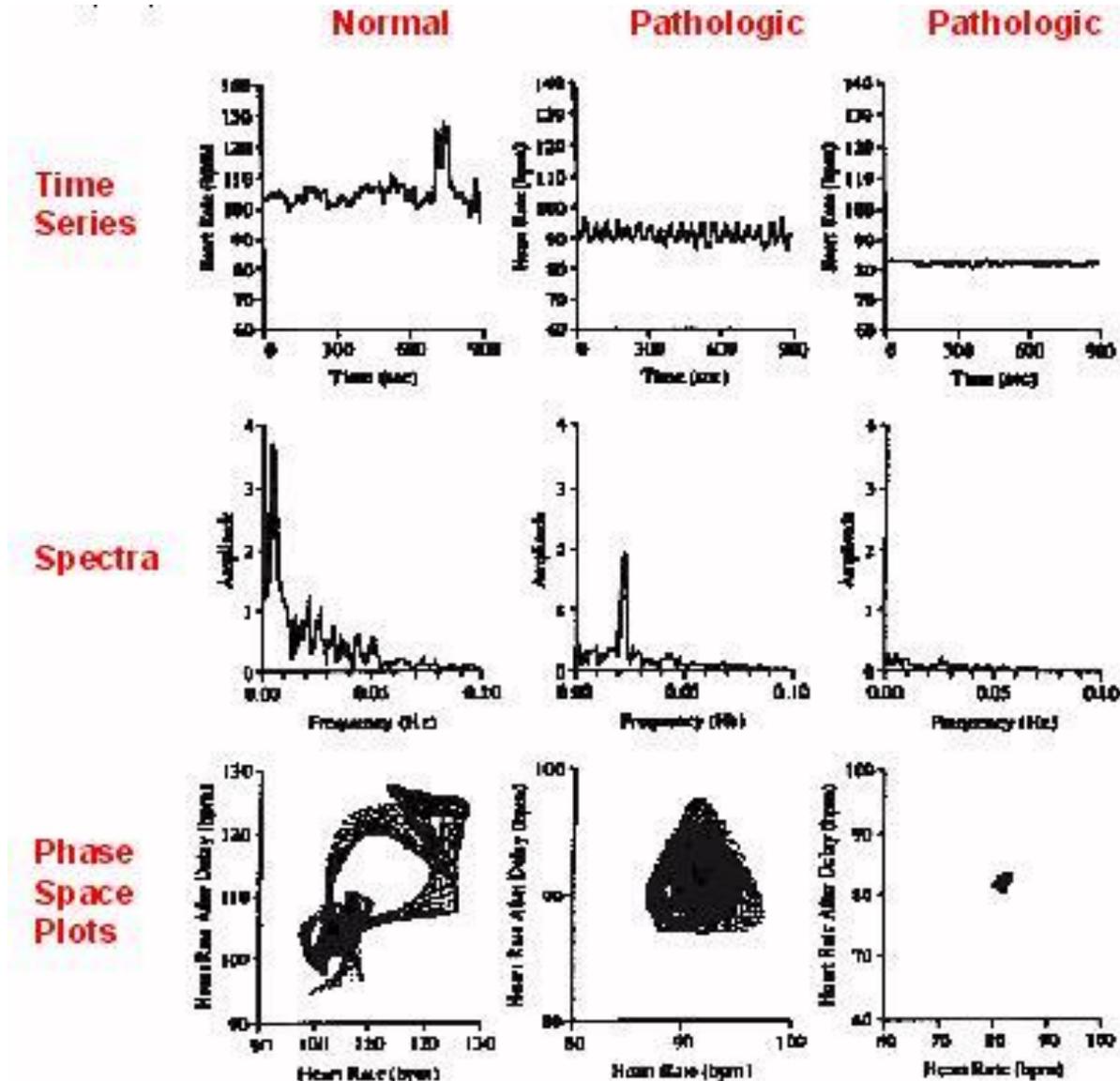


**EEG snimak**



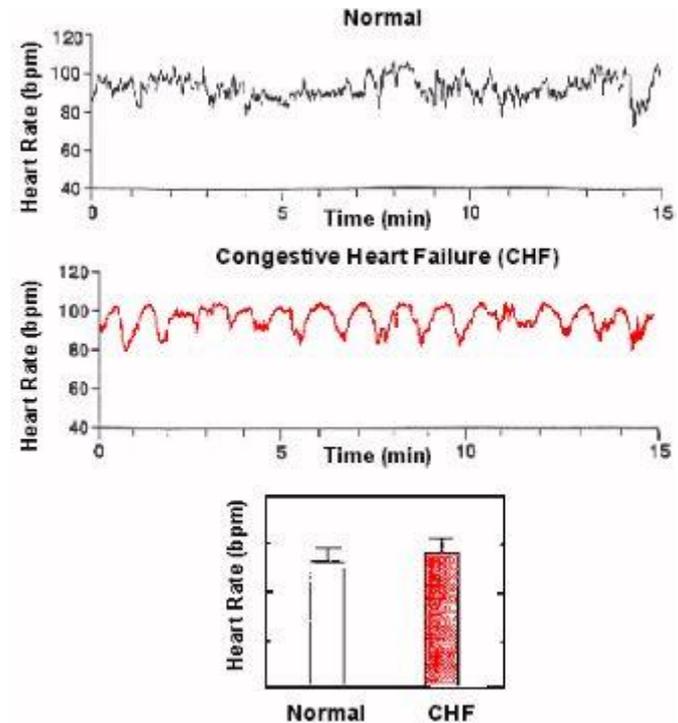
**Neuronov atraktor**

# Dinamika rada srca



# Kontrola haosa

- 1. Ventrikularna fibrilacija I odgovarajuca tahikardija koje izazivaju iznenadnu smrt su priblizno periodicna kretanja, a ne haoticni proces!
- 2. Za razliku, zdravo srce pokazuje haoticno kretanje. Iznenadna smrt se s toga moze smatrati kao bifurkacija iz haosa, a ne u haos.



# Zaključak

- Teorija nelinearnih oscilacija i haosa objasnjavaju neke fenomene u prirodi ali i u tehnici.
- Manifestacije u realnim sistemima zahtevaju proširenje našeg saznanja o nelinearnim sistemima.
- To će nam dati mogućnost tačnijeg predviđanja.